Multivariable Integral Guide

Integral	Notation	Application

Basic Integrals — over "flat" regions, evaluated as iterated integrals

$\int_a^b f(x)\,dx$		Area under curve; Average value of f on $[a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx$; density=$\rho(x)$, Mass= $\int_a^b \rho(x)\,dx$; velocity= $v(t)$, distance traveled= $\int_a^b v(t)\,dt$; etc.
$\iint_D f(x,y)\,dA$	$dA = dx\,dy$ $= r\,dr\,d\theta$	Volume under surface; Area of $D = \iint_D dA$ Average value of f on $D = \frac{1}{\text{Area of } D}\iint_D f(x,y)\,dA$; $\rho(x,y)$ =density, Mass= $\iint_D \rho(x,y)\,dA$; etc.
$\iiint_R f(x,y,z)\,dV$	$dV = dx\,dy\,dz$ $= r\,dz\,dr\,d\theta$ $= \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$	Volume of $R = \iiint_R dV$ Average value of f on $R = \frac{1}{\text{Volume of } R}\iiint_R f(x,y,z)\,dV$; $\rho(x,y,z)$ =density, Mass= $\iiint_R \rho(x,y,z)\,dV$; etc.

Integrals of **scalar functions** over "curved" things — require parameterizations, to become iterated integrals

$\int_C f(x,y)ds$, $\int_C f(x,y,z)ds$	$\mathbf{r}(t)$ parameterizes curve C $ds = \lvert\lvert\mathbf{r}'(t)\rvert\rvert dt$	Length of $C = \int_C ds$; Average value of f on $C = \frac{1}{\text{Length of C}}\int_C f, ds$
$\iint_S f(x,y,z)\,dS$	$\Phi(u,v)$ parameterizes surface S $T_u = \frac{\partial}{\partial u}\Phi, T_v = \frac{\partial}{\partial v}\Phi$ $dS = \lvert\lvert T_u \times T_v\rvert\rvert du\,dv$	Surface area of $S = \iint_S dS$; Average value of f on $S = \frac{1}{\text{Area of } S}\iint_S f(x,y,z)\,dS$

Integrals of **vector fields** over "curved" things — require parameterizations to become iterated integrals

$\int_C F(x,y)\cdot d\mathbf{s}$ $= \int_C P\,dx + Q\,dy$, $\int_C F(x,y,z)\cdot d\mathbf{s}$ $= \int_C Pdx + Qdy + Rdz$	$\mathbf{r}(t)$ parameterizes curve C $d\mathbf{s} = \mathbf{r}'(t)dt$	Work (F is force); Circulation (F is velocity, C is a loop)
$\iint_S F(x,y,z)\cdot d\mathbf{S}$	$\Phi(u,v)$ parameterizes surface S $T_u = \frac{\partial}{\partial u}\Phi, T_v = \frac{\partial}{\partial v}\Phi$ $d\mathbf{S} = T_u \times T_v\,du\,dv$	Flux of F through S

Theorems relating integrals and derivatives — general form: $\displaystyle\iint_B \partial F = \int_{\partial B} F$

Name	Statement
Fundamental Theorem of Calculus (in $\mathbb{R}$)	$\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$
Green's Theorem (in $\mathbb{R}^2$)	$\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{C=\partial D} P\,dx + Q\,dy$
Stokes' theorem (in $\mathbb{R}^3$)	$\displaystyle\iint_S (\nabla \times F)\cdot d\mathbf{S} = \oint_{\partial S} F\cdot d\mathbf{S}$
Gauss' Divergence Theorem (in $\mathbb{R}^3$)	$\displaystyle\iiint_R (\nabla \cdot F)\; dV = \iint_{S=\partial R} F\cdot d\mathbf{S}$