Final Exam
Math 202 F01

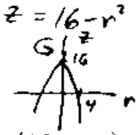
Name: Solutions
12/15/05

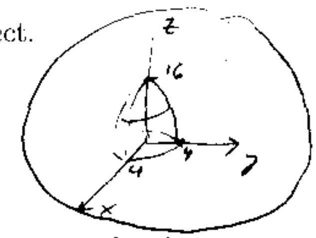
1. An object fills the solid region between the graph of $z = 16 - x^2 - y^2$ and the plane $z = 0$. The density at any point is given by $f(x, y, z) = 2z$. Here x, y, z are measured in cm, and f is in g/cm^3.

   (a) (3 pts.) Sketch the object.

   $z = 16 - x^2 - y^2$

   $z = 16 - r^2$

   (b) (10 pts.) Compute the mass of the object.

   $$\iiint_R z\,dV = \int_0^{2\pi}\int_0^4\int_0^{16-r^2} 2z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^4 z^2\Big|_0^{16-r^2} r\,dr\,d\theta$$

   $$= \int_0^{2\pi}\int_0^4 (16-r^2)^2 r\,dr\,d\theta = -\frac{1}{2}\int_0^{2\pi}\int_{16}^{0} u^2\,du\,d\theta$$

   $u = 16 - r^2$
   $du = -2r\,dr$

   $$= -\frac{1}{2}\int_0^{2\pi} \frac{u^3}{3}\Big|_{16}^{0} d\theta = \frac{1}{2}\int_0^{2\pi}\frac{16^3}{3}\,d\theta = \left(\frac{1}{2}\right)\frac{16^3}{3}(2\pi)$$

   $$= \frac{\pi(16)^3}{3}$$

2. (6 pts.) For points near $(4, 1)$, give a linear approximation to the function

$$f(x, y) = xy^2 - 3\sqrt{xy}.$$

$f(4,1) = 4 - 6 = -2$

$\frac{\partial f}{\partial x}(4,1) = y^2 - \frac{3}{2}\frac{\sqrt{y}}{\sqrt{x}}\Big|_{(4,1)} = 1 - \frac{3}{4} = \frac{1}{4}$

$\frac{\partial f}{\partial y}(4,1) = 2xy - \frac{3}{2}\frac{\sqrt{x}}{\sqrt{y}}\Big|_{(4,1)} = 8 - 3 = 5$

so $L(x,y) = -2 + \frac{1}{4}(x-4) + 5(y-1)$

3. (6 pts.) What is the directional derivative of $f(x, y, z) = 2xy - z^2$ at the point $(1, 2, 1)$ in the direction toward the origin?

$\nabla f\big|_{(1,2,1)} = (2y, 2x, -2z)\big|_{(1,2,1)} = (4, 2, -2)$

$\vec{u} = \frac{-(1,2,1)}{\|(1,2,1)\|} = -\frac{1}{\sqrt{6}}(1, 2, 1)$ $\quad \nabla f\big|_{(1,2,1)} \cdot \vec{u} = -\frac{1}{\sqrt{6}}(4,2,-2)\cdot(1,2,1) = \frac{-6}{\sqrt{6}}$

$= -\sqrt{6}$

4. Consider the vector field $F(x, y, z) = (2xy + z^2, x^2 + z - 1, 2 + y + 2xz)$.

   (a) (7 pts.) Either find a potential function for F, or show one does not exist.

   $\frac{\partial f}{\partial x} = 2xy + z^2$

   so $f(x,y,z) = x^2y + xz^2 + C(y,z)$

   so $\frac{\partial f}{\partial y} = x^2 + \frac{\partial C}{\partial y}(y,z) = x^2 + z - 1 \Rightarrow \frac{\partial C}{\partial y}(y,z) = z - 1, \ C(y,z) = yz - y + D$

   so $f(x,y,z) = x^2y + xz^2 + yz - y + D(z)$

   so $\frac{\partial f}{\partial z} = 2xz + y + \frac{dD}{dz} = 2 + y + 2xz \Rightarrow \frac{dD}{dz} = 2, \ D(z) = 2z + E$

   so $f(x,y,z) = x^2y + xz^2 + yz - y + 2z + E$

   (b) (7 pts.) Supposing F represents a force field, compute the amount of work done by F on a particle moving along the path

$$\mathbf{r}(t) = (1 + t, t^2, t^3 - 4t), \quad 0 \le t \le 2.$$

   (You may find it helpful to use your answer to part (a).)

   $W = \int_C F \cdot d\vec{r} = f\left(\begin{smallmatrix}\text{end}\\ \text{of}\\ C\end{smallmatrix}\right) - f\left(\begin{smallmatrix}\text{start}\\ \text{of}\\ C\end{smallmatrix}\right)$

   $= f(\vec{r}(2)) - f(\vec{r}(0))$

   $= f(3, 4, 0) - f(1, 0, 0)$ but using f from (a)

   $= (3^2 \cdot 4 - 4) - 0 = 32$

5. (15 pts. — 3 pts. each) Consider the point $\mathbf{p} = (1, 2, -1)$, and the two vectors $\mathbf{v} = (1, 0, -1)$, $\mathbf{w} = (2, -2, 2)$.

(a) Give a parameterization of the line through $\mathbf{p}$ with direction $\mathbf{w}$.

$$\vec{r}(t) = (1,2,-1) + t\,(2,-2,2)$$
$$= (1+2t,\ 2-2t,\ -1+2t)$$

(b) Find a vector orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

$$\vec{V} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 2 & -2 & 2 \end{vmatrix} = (-2,-4,-2)$$

(or any multiple, such as $(1,2,1)$)

(c) Give an equation of the plane through $\mathbf{p}$ containing the directions $\mathbf{v}, \mathbf{w}$.

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \qquad 1\cdot(x-1) + 2(y-2) + 1(z-{-1}) = 0$$
$$x + 2y + z = 4$$

(d) Give a parameterization of the plane through $\mathbf{p}$ containing the directions $\mathbf{v}, \mathbf{w}$.

$$\vec{r}(s,t) = (1,2,-1) + s\,(1,0,-1) + t\,(2,-2,2)$$
$$\vec{r}(s,t) = (1+s+2t,\ 2-2t,\ -1-s+2t)$$

$\underline{\underline{\text{or}}}$ $\vec{r}(s,t) = (s,\ t,\ 4-s-2t)$ (using part (c))

(e) Find the angle between $\mathbf{v}$ and $\mathbf{w}$.

$$\vec{V} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta$$

But $\vec{v} \cdot \vec{w} = (1,0,-1) \cdot (2,-2,2) = 0 \Rightarrow \theta = \pi/2$ $(90°)$

6. (10 pts.) A surface S is parameterized by

$$\mathbf{r}(u,v) = (u\cos v, u, u\sin v), \quad 1 \le u \le 2,\ 0 \le v \le \pi$$

Set up an integral to compute the flux through S of the vector field $F(x,y,z) = (x,1,y)$. *Do not evaluate the integral, but leave it in a form where that is all that remains to be done.*

$$\iint_S F \cdot d\vec{S} = \int_0^{\pi}\int_1^2 (u\cos v, 1, u)\cdot(u\cos v, -u, u\sin v)\,du\,dv$$

$$T_u = (\cos v, 1, \sin v)$$

$$T_v = (-u\sin v, 0, u\cos v)$$

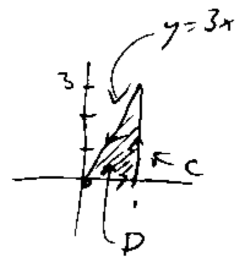
$$T_u \times T_v = (u\cos v, -u, u\sin v)$$

$$= \int_0^{\pi}\int_1^2 (u^2\cos^2 v - u + u^2\sin v)\,du\,dv$$

7. (10 pts.) Use Green's theorem to evaluate the integral

$$\oint_C x^2y^2\,dx - 2x^3y\,dy$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$.



Since $\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$

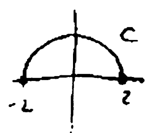
$$\oint_C x^2y^2\,dx - 2x^3y\,dy = \iint_D (-6x^2y - 2x^2y)\,dy\,dx$$

$$= \int_0^1\int_0^{3x} -8x^2y\,dy\,dx = \int_0^1 -4x^2y^2\Big|_0^{3x}\,dx$$

$$= \int_0^1 -36x^4\,dx = -\frac{36}{5}x^5\Big|_0^1 = -\frac{36}{5}$$

4

8. (10 pts.) A wire, with uniform density, is shaped like a semicircle around the origin, going from $(2, 0)$ to $(-2, 0)$ in the plane. Find the coordinates of the centroid of the wire (i.e., of the center of mass).



By symmetry, $\bar{x} = 0$

$r(t) = (2\cos t, 2\sin t)$

$r'(t) = (-2\sin t, 2\cos t)$

$\|r'(t)\| = 2$

$0 \le t \le \pi$

$$\bar{y} = \frac{\int_C y\,ds}{\int_C ds} = \frac{\int_0^\pi (2\sin t)\,2\,dt}{\text{length of } C} = \frac{4\int_0^\pi \sin t\,dt}{\frac{1}{2}(2\pi 2)} = \frac{8}{2\pi} = \frac{4}{\pi}$$

9. (16 pts. — 2 pts. each) The following may or may not be correct. If they are, mark them 'True.' If they are not, give corrected versions.

(a) In spherical coordinates, $dV = \cancel{\rho^2 \cos\phi\, d\rho}\, d\phi\, d\theta$.

$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

(b) In cylindrical coordinates, $dV = r\, dz\, d\theta\, dr$.

True

(c) The area of a parallelogram with sides given by vectors $\mathbf{v}$ and $\mathbf{w}$ is $\cancel{\mathbf{v} \cdot \mathbf{w}}$.

$\|\vec{v} \times \vec{w}\|$

(d) If $\nabla f(1,-2) = (0,0)$, then f must have a local maximum or a minimum at $(1,-2)$, or a saddle point, or other possibilities

$(1,-2)$ is a critical pt, but that just means it is a candidate for a max/min

(e) If $-C$ denotes 'C backwards,' then $\cancel{\int_C F \cdot d\mathbf{r} = \int_{-C}} F \cdot d\mathbf{r}$.

$$\int_C F \cdot d\vec{r} = -\int_{-C} F \cdot d\vec{r}$$

(f) If $-C$ denotes 'C backwards,' then $\int_C f\,ds = \int_{-C} f\,ds$.

True

(g) The main reason its useful to know how to compute integrals is ~~to be able to find areas.~~ many applications require "adding up little pieces of something". E.g., work, flux, center of mass, mass, etc.

(h) If div $F(1,-1,2) = (1,0,0)$, then the vector field F has a net flow out from the point $(1,-1,2)$ in the direction given by $(1,0,0)$.

div F is not a vector. One corrected version could be:

If div $F(1,-1,2) = 1$, then F has a net flow out from the point $(1,-1,2)$