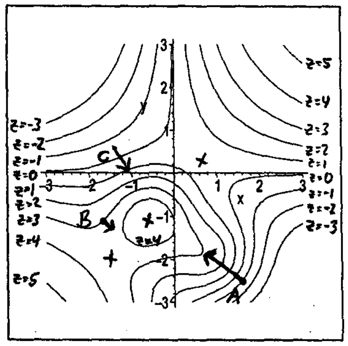
Exam 2
Math 202 F01

Name: Solutions

11/11/05

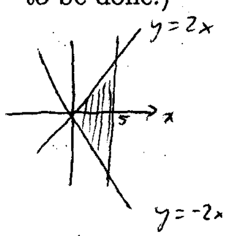
1. (10 pts.) A plot of level curves of a function $z = f(x, y)$ is given:



(a) At the points A, B, C, draw the gradient for f. You should be able to indicate the correct direction, and correct *relative* lengths of the three. Note $\|\nabla f(A)\| > \|\nabla f(C)\| > \|\nabla f(B)\|$

(b) Mark with an 'X' all points where the gradient is $\vec{0}$

2. (16 pts.) A metal plate is shaped like the region between the three lines $y = \pm 2x$ and $x = 5$, where x and y are measured in m. The density of the plate is given by $\rho(x, y) = 1+x\ kg/m^2$. Give an expression involving integrals for the center of mass of the plate. (Do NOT evaluate any integrals, but leave your answer in a form where that is all that remains to be done.)



$$\text{mass} = m = \int_0^5 \int_{-2x}^{2x} (1+x)\, dy\, dx$$

$$\bar{x} = \frac{\int_0^5 \int_{-2x}^{2x} x(1+x)\, dy\, dx}{m}$$

$$\bar{y} = \frac{\int_0^5 \int_{-2x}^{2x} y(1+x)\, dy\, dx}{m} = 0 \quad \left(\text{by symmetry of region} + \rho(x,y)\right)$$

3. (14 pts.) Give an equation of the tangent plane to the surface

$$xy + yz^2 + zx = -1$$

at the point $(1, 0, -1)$.

Let $g(x,y,z) = xy + yz^2 + zx$

$\nabla g = (y+z,\ x+z^2,\ 2yz+x)$

$\nabla g(1,0,-1) = (-1, 2, 1)$

So plane is $-1(x-1) + 2(y-0) + 1(z--1) = 0$

or $-x + 2y + z = -2$

4. The temperature at a point (x, y) is given by

$$T = \frac{y}{1+x^2} + z^2.$$

(a) (10 pts.) Compute the directional derivative of temperature at the point $(2, 1, 1)$ in the direction towards (3,2,1).

$\nabla T = \left(\frac{-2xy}{(1+x^2)^2}, \frac{1}{1+x^2}, 2z\right)$

$\nabla T(2,1,1) = \left(\frac{-4}{25}, \frac{1}{5}, 2\right)$

$(3,2,1) - (2,1,1) = (1,1,0)$, so $\vec{u} = \frac{(1,1,0)}{\sqrt{2}}$

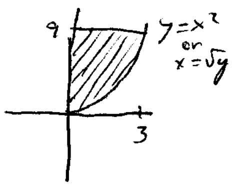
$D_{\vec{u}}T = \nabla T \cdot \vec{u} = \left(\frac{-4}{25}, \frac{1}{5}, 2\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = -\frac{4}{25\sqrt{2}} + \frac{1}{5\sqrt{2}} = \frac{1}{25\sqrt{2}}$

(b) (6 pts.) At the point $(2, 1, 1)$, in what direction should you travel in order to get the fastest *increase* in temperature?

In direction given by $\nabla T(2,1,1) = \left(\frac{-4}{25}, \frac{1}{5}, 2\right)$

5. (28 pts. – 14 pts. each) Evaluate the following integrals by first following the suggestions.

(a) $\int_0^3 \int_{x^2}^9 x\cos(y^2)\, dy\, dx$ (Reverse the order of integration.)



$$= \int_0^9 \int_0^{\sqrt{y}} x \cos(y^2)\, dx\, dy$$

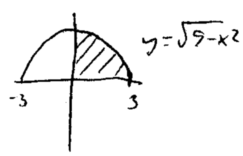
$$= \int_0^9 \frac{x^2}{2}\cos(y^2)\Big|_{x=0}^{\sqrt{y}} dy = \frac{1}{2}\int_0^9 y\cos(y^2)\, dy$$

$u = y^2$

$du = 2y\,dy$

$$= \frac{1}{2}\int_0^{81} \frac{1}{2}\cos(u)\, du = \frac{1}{4}\sin(u)\Big|_0^{81} = \frac{\sin 81}{4}$$

(b) $\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{(10-x^2-y^2)}\, dy\, dx$ (Use a different coordinate system.)



$$= \int_0^{\pi/2} \int_0^3 \sqrt{10-r^2}\; r\, dr\, d\theta$$

$u = 10 - r^2$

$du = -2r\,dr$

$$= \int_0^{\pi/2} \int_{10}^{1} -\frac{1}{2}\sqrt{u}\, du\, d\theta = \int_0^{\pi/2} -\frac{1}{3}u^{3/2}\Big|_{10}^{1} d\theta$$

$$= \int_0^{\pi/2} \left(-\frac{1}{3} + \frac{10^{3/2}}{3}\right) d\theta = \frac{\pi}{2}\left(\frac{10^{3/2}}{3} - \frac{1}{3}\right)$$

$$= \frac{\pi}{6}\left(10^{3/2} - 1\right)$$

6. (16 pts.) Find the maximum of the function $f(x, y, z) = 8x - 4z$, subject to the constraint that $x^2 + 10y^2 + z^2 = 5$

Let $g(x,y,z) = x^2 + 10y^2 + z^2$

$\nabla f = (8, 0, -4)$

$\nabla g = (2x, 20y, 2z)$

So $\nabla f = \lambda \nabla g$ means

$$\left.\begin{array}{ll} 1) & 8 = 2x\lambda \\ 2) & 0 = 2y\lambda \\ 3) & -4 = 2z\lambda \\ 4) & x^2 + 10y^2 + z^2 = 5 \end{array}\right\} \text{4 equations in 4 unknowns}$$

Equations 1+3 indicate $\lambda \neq 0$

+ using this in equation 2 gives $y = 0$

Then 1+3 can be expressed $x = \frac{4}{\lambda}$

$z = -2/\lambda$

Plugging all those into equation 4 gives

$$\left(\frac{4}{\lambda}\right)^2 + 10(0)^2 + \left(\frac{-2}{\lambda}\right)^2 = 5$$

$$\frac{16}{\lambda^2} + \frac{4}{\lambda^2} = 5$$

$$20 = 5\lambda^2$$

$$\lambda^2 = 4$$

$$\lambda = \pm 2$$

$\lambda = 2 \Rightarrow (x,y,z) = (2,0,-1)$ $\quad f(2,0,-1) = 8 \cdot 2 - 4(-1) = 20$

$\lambda = -2 \Rightarrow (x,y,z) = (-2,0,1)$ $\quad f(-2,0,1) = 8(-2) - 4(1) = -20$

So max is