

## On convergence of the explicit method for the general linear parabolic equation (2.140)

In class on Wednesday 2 March I got myself slightly confused when I allowed a nonnegative coefficient  $c(x, t)$  in the general linear heat-like (“parabolic”) equation from section 2.15:

$$u_t = a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u + d(x, t). \quad (2.140)$$

As stated in the text, as long as  $c(x, t)$  is bounded above, this is a well-posed continuum problem. Furthermore, there will be conditional stability for the explicit numerical method, as I show below.

Exponential growth of the solution to (2.140) is possible. For example, the special case  $u_t = c(x)u$  has exact solution  $u(x, t) = e^{c(x)t}u_0(x)$  if  $u(x, 0) = u_0(x)$ , so the solution grows exponentially where  $c(x) > 0$ . The mathematical model therefore includes exponential growth and so the numerical scheme must approximate it. On the other hand, if there is exponential growth at a rate *which becomes infinite as  $\Delta t \rightarrow 0$*  then we have *numerical instability*.

In section 2.15 we have equation (2.142) for the error  $e_j^n := U_j^n - u(x_j, t_n)$  for the explicit method on (2.140). Defining  $\nu = \Delta t / \Delta x^2$  and  $\mu = \Delta t / \Delta x$  we see that  $e_j^n$  satisfies

$$e_j^{n+1} = (1 - 2\nu a_j^n + \Delta t c_j^n)e_j^n + (\nu a_j^n + \frac{1}{2}\mu b_j^n)e_{j+1}^n + (\nu a_j^n - \frac{1}{2}\mu b_j^n)e_{j-1}^n - \Delta t T_j^n \quad (2.142)$$

We require  $a(x, t) > 0$  for well-posedness. The coefficients are nonnegative in (2.142) if the two conditions (2.144):  $\frac{1}{2}\mu|b_j^n| \leq \nu a_j^n$  and (2.145):  $2\nu a_j^n - \Delta t c_j^n \leq 1$  apply.

I can now get to the point. If indeed  $c(x, t) \leq C$  for some  $C$ , and if conditions (2.144) and (2.145) apply, then the coefficients on the right side of (2.142) are nonnegative and sum to  $1 + \Delta t c_j^n$ . Thus if  $E^n = \max_j |e_j^n|$  then  $E^0 = 0$  and

$$E^{n+1} \leq \max_j (1 + \Delta t c_j^n) E^n + \Delta t \max_j |T_j^n| \leq (1 + C\Delta t) E^n + T\Delta t$$

where  $T = \max_{j,n} |T_j^n|$ . (Note that the text does state this inequality.) It follows that

$$\begin{aligned} E^n &\leq (1 + C\Delta t)^{n-1} T\Delta t + \cdots + (1 + C\Delta t)^2 T\Delta t + (1 + C\Delta t) T\Delta t + T\Delta t \\ &= \left( \sum_{k=1}^{n-1} (1 + C\Delta t)^k \right) T\Delta t = \frac{1 - (1 + C\Delta t)^n}{1 - (1 + C\Delta t)} T\Delta t = \frac{(1 + C\Delta t)^n - 1}{C\Delta t} T\Delta t \\ &\leq \frac{(1 + Ct_f/n)^n - 1}{C\Delta t} T\Delta t \leq \frac{e^{Ct_f} - 1}{C\Delta t} T\Delta t = \frac{e^{Ct_f} - 1}{C} T. \end{aligned}$$

I have summed a geometric series. The last two inequalities are worth commenting on. Recall that  $n\Delta t \leq t_f$  and that  $(1 + \frac{\alpha}{n})^n \rightarrow e^\alpha$  as  $n \rightarrow \infty$ . (The limit is approached from below if  $\alpha \geq 0$ .)

If  $c(x, t) \leq 0$  the analysis is greatly simplified:  $E^{n+1} \leq E^n + T\Delta t$  so  $E^n \leq t_f T$ .

Thus we know that  $E^n \leq KT$  for all  $n$  such that  $n\Delta t \leq t_f$ , where  $K = (e^{Ct_f} - 1)/C$  if  $C > 0$  and  $K = t_f$  if  $C \leq 0$ . But as  $\Delta t, \Delta x \rightarrow 0$  we have  $T \rightarrow 0$  because the explicit method is consistent. Thus we have convergence (and stability).