Review Topics for Midterm Quiz
on Wednesday 25 March

Definitions. Know how to define:

- matrix-vector product; matrix-matrix product
- adjoint; hermitian; transpose; symmetric
- inner product
- unitary
- $\|\cdot\|_p$ of a vector for any $1 \le p \le \infty$
- induced matrix norm $\|\cdot\|$
- projector; orthogonal projector
- eigenvalue; eigenvector; singular value
- residual (for overdetermined "$Ax = b$")
- absolute condition number $\hat{\kappa}(x)$ of a problem instance
- relative condition number $\kappa(x)$ of a problem instance
- condition number of a square matrix: $\kappa(A) = \|A\|\,\|A^{-1}\|$

Matrix Factorizations and Constructions. Here $A \in \mathbb{C}^{m\times n}$. Know the properties of the factors (e.g. " $\hat{Q}$ has orthonormal columns and is of same size as A in reduced QR factorization $A = \hat{Q}\hat{R}$"), and be able to use the factorization in the simplest calculations:

- full SVD: $A = U\Sigma V^*$
- reduced SVD, $m \ge n$: $A = \hat{U}\hat{\Sigma}V^*$
- full QR, $m \ge n$: $A = QR$
- reduced QR, $m \ge n$: $A = \hat{Q}\hat{R}$
- eigenvalue, $m = n$: $AX = X\Lambda$
- orthogonal projector, $\hat{Q}$ has $m \ge n$: $P = \hat{Q}\hat{Q}^*$
- orthogonal projector, A full rank with $m \ge n$: $P = A\left(A^*A\right)^{-1}A^*$
- Householder reflector: $F = I - 2vv^*/(v^*v)$

Algorithms. Be able to summarize the algorithms, including the amount of work:

- classical Gram-Schmidt for QR
- modified Gram-Schmidt for QR (informal knowledge o.k.)
- Householder triangularization for QR
- QR "modern classical" for least squares (for overdetermined "$Ax = b$")

Formulas and Inequalities.

- 1-norm of a matrix as the maximum column sum
- ∞-norm of a matrix as the maximum row sum
- $\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2} = \sqrt{\operatorname{tr}(A^*A)}$
- $\|A\|_2 = \sigma_1 = \sqrt{\rho(A^*A)}$
- Cauchy-Schwarz: $|x^*y| \le \|x\|\,\|y\|$
- invariance of $\|\cdot\|_2$ and $\|\cdot\|_F$ matrix norms under unitaries
- $\operatorname{rank}(A)$ is number of nonzero singular values
- if $m = n$, $|\det(A)| = \prod_{i=1}^m \sigma_i$
- if P is a projector then so is $I - P$
- if P is an orthogonal projector then so is $I - P$, and furthermore $I - 2P$ is unitary
- if A is square and $\|\cdot\| = \|\cdot\|_2$: $\quad \kappa(A) = \sigma_1/\sigma_m$

Other. Ideas to be comfortable with; in some cases these are provable theorems but in other cases there is just a perspective to "get":

- how to think about Ax, $A^{-1}b$, Qx, Q^*b
- the image of the unit sphere under any $m \times n$ matrix is a hyperellipse
- sums like this are optimal (in what sense?) approximators of A, among rank ν matrices:
$$A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^*$$
- if $A \in \mathbb{C}^{m\times n}$ then A has full rank if and only if A^*A is nonsingular
- construction of normal equations $A^*Ax = A^*b$ to solved overdetermined "$Ax = b$" if $A \in \mathbb{C}^{m\times n}$ and $m > n$
- construction of orthogonal functions (e.g. orthogonal polynomials) is instance of Gram-Schmidt and of "QR" where columns are infinitely long
- the number of operations in an algorithm can be counted effectively (quantitative meaning?) by counting only number of times inner-most operations are executed
- the number of operations in an algorithm can be counted by geometric arguments
- Gram-Schmidt is "triangular orthogonalization" while Householder is "orthogonal triangularization"