Midterm Quiz: Solutions

1. **(a)** The *column rank* is the dimension of the column space (the span of the columns, a subset of $\mathbb{C}^m$). The *row rank* is the dimension of the row space (the span of the rows, a subspace of $\mathbb{C}^n$). It is a theorem that the column and row ranks are the same, and define the *rank* of A. So A has *full rank* if the rank of A is equal to the minimum of m or n, which is the maximum it can be for a matrix of that size.
(b) *Sorry about this. I forgot it was on Assignment # 5; see solutions for that. Here you are only to prove one direction, but the proof is the SVD either way ...*

2. *And sorry about this one too! See solutions to Assignment #5.*

3. One possibility, the simplest, is:

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

4. **(a)** $\|A\|_\infty = (\text{max. abs. row sum}) = 7$,
$\|A\|_1 = (\text{max. abs. column sum}) = 6$,
$\|A\|_F = (\sum_{ij} a_{ij}^2)^{1/2} = 4\sqrt{2}$.
(b) We need the largest singular value because $\|A\|_2 = \sigma_1$. So: Form A^*A. Find its eigenvalues by forming the characteristic polynomial $p(\lambda) = \det(A^*A - \lambda I)$. This polynomial will only have positive, real roots because A^*A is symmetric and positive definite. Find the largest root $\lambda_1 = \sigma_1^2$, and take the square root of that to give $\|A\|_2$.

5.

$$r_{ij} = \begin{cases} 0, & i > j \quad \text{(below diagonal)}, \\ q_i^* a_j, & i < j \quad \text{(above diagonal)}, \\ \left\| a_j - \sum_{k=1}^{j-1} r_{kj} q_k \right\|, & i = j. \end{cases}$$

6. Since $\kappa(A) = \|A\|_1 \|A^{-1}\|_2$ in this case, and assuming $A = QR$ for Q unitary, we have

$$\kappa(A) = \|QR\|_2 \|(QR)^{-1}\|_2 = \|QR\|_2 \|R^{-1}Q^*\|_2 = \|R\|_2 \|R^{-1}\|_2.$$

Note that $Q^{-1} = Q^*$ is unitary if Q is unitary.

The key here is that the 2-norm is invariant under unitary multiplication: $\|Q_1 B\|_2 = \|BQ_2\|_2 = \|B\|_2$ *for any matrix* B *and any* Q_1, Q_2 *unitary of the correct dimensions so that the product with* B *makes sense.*

I was curious to see if this is how `cond` *in* MATLAB | OCTAVE *is computed, but the* OCTAVE *implementation uses the SVD, at least. Recall* $\kappa(A) = \sigma_1/\sigma_m$. *In retrospect the SVD implementation makes sense. Once you factor* $A = QR$ *you still need to compute the norms* $\|R\|_2$, $\|R^{-1}\|_2$. *Computing the 2-norm essentially requires the SVD anyway, as in* **4(b)** *above. In any case,* R^*R *is not triangular anyway, so its 2-norm is essentially as hard to compute as that of* A.

7. **(b)**

$$(\text{total number of ops}) = 1 + \sum_{j=2}^{m} \left(\left(\sum_{k=1}^{j-1} 2 \right) + 1 \right) = 1 + (m-1) + \sum_{j=2}^{m} 2(j-1) = m + 2\sum_{j=2}^{m}(j-1)$$

$$= m + 2\sum_{l=1}^{m-1} l = m + 2\,\frac{m(m-1)}{2} = m + m(m-1) = m^2.$$

Exactly. So the total number of operations is $O(m^2)$, which has $C = 1$ and $r = 2$.

Back substitution, which we will actually use, has the same operations count.