

Name: \_\_\_\_\_

Math 694 Numerical Linear Algebra (Bueler)

24 March 2009

## Midterm Quiz on Fundamentals

(45 points possible)

1. (a) (5 pts) Define the *column rank* and *row rank* of  $A \in \mathbb{C}^{m \times n}$ . Also, define what it means for  $A$  to have *full rank*.

(b) (5 pts) Show that if  $A \in \mathbb{C}^{m \times n}$  has full rank then  $A^*A$  is nonsingular. (Hint: *You may use the SVD.*)

**2.** (5 pts) Show that if  $P$  is an orthogonal projector then  $I - 2P$  is unitary.

**3.** (5 pts) Compute the full SVD of

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 3 & 0 \end{bmatrix}.$$

4. (a) (5 pts) Compute  $\|A\|_\infty$ ,  $\|A\|_1$ , and  $\|A\|_F$  if

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 1 \\ 3 & 0 & 4 \end{bmatrix}.$$

- (b) (5 pts) Describe how you would compute  $\|A\|_2$  by hand, e.g. on  $A$  in part (a).

**5.** (5 pts) Suppose you do Gram-Schmidt orthogonalization on a set of vectors  $\{a_1, a_2, \dots, a_n\}$  in  $\mathbb{C}^m$ . This gives an orthonormal set  $\{q_1, q_2, \dots, q_n\}$  with the property that for  $j = 1, \dots, n$  the set  $\{q_1, q_2, \dots, q_j\}$  is an orthonormal basis of  $\{a_1, a_2, \dots, a_j\}$ . Give formulas for numbers the  $r_{ij}$  so that  $a_j = r_{1j}q_1 + \dots + r_{jj}q_j$ . That is, explain the core of the classical Gram-Schmidt algorithm as a method for  $QR$  factorization.

**6.** (5 pts) Suppose  $A \in \mathbb{C}^{m \times m}$  is invertible (nonsingular) and that  $A = QR$  is a (full) QR decomposition. Show that, for the 2-norm,

$$\kappa(A) = \|R\| \|R^{-1}\|.$$

7. Suppose  $L \in \mathbb{C}^{m \times m}$  is lower triangular, so  $L_{ij} = 0$  if  $i < j$ , and assume  $b \in \mathbb{C}^m$ . The following algorithm solves  $Lx = b$  in the obvious way, by *forward substitution*.

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 $x_1 = b_1/L_{11}$ 
for  $j = 2$  to  $m$ 
   $a = b_j$ 
  for  $k = 1$  to  $j - 1$ 
     $a = a - L_{jk}x_k$ 
   $x_j = a/L_{jj}$ 

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(a) (0 pts; for your benefit) Confirm that you understand this algorithm by using it to solve

$$\begin{aligned} L_{11}x_1 &= b_1 \\ L_{21}x_1 + L_{22}x_2 &= b_2 \\ L_{31}x_1 + L_{32}x_2 + L_{33}x_3 &= b_3 \end{aligned}$$

For example you might write out explicit formulas for  $x_1, x_2, x_3$ . (But I will not grade your response, regardless.)

(b) (5 pts) For arbitrary  $m$ , count the exact total number of operations in the above algorithm, counting each addition, subtraction, multiplication, and division equally as one operation. Also give the leading order estimate, in form “ $O(Cm^r)$ ” for specific  $C$  and  $r$ .