Assignment # 7

Due Friday 10 April at start of class

Exercise 12.1.

Exercise 12.2.

Exercise 13.3. *I will treat* 13.3 *in the text as having a typo. In any case, redefine part* (a) *as follows, so all inputs x are near* 2, *using* MATLAB *notation*:

(a) Plot $p(x)$ for $x = 1.920 : 0.001 : 2.080$, evaluating p via its coefficients $1, -18, 144, \dots$

Part (b) *is unchanged.*

Exercise 14.2.

Exercise 15.1abc.

P15. This problem could be titled,

Where do circulant matrices come from, and where do least squares problems come from?

This is a deliberately easy problem, since you have already done hard stuff related to both circulant and least squares. (There are a lot of words, but student actions are brief.)

Consider a differential equations problem, of the same type Fourier himself considered. Consider a ring of circumference $L = 1$, and spatial variable x so $0 \le x \le 1$ but $x = 0$ and $x = 1$ actually describe the same point. Let the temperature be $u(x)$. Suppose some distribution of heat sources $f(x)$ and with heat loss proportional to temperature (and proportionality constant picked without any motivation to be "3"):

$$0 = u_{xx} - 3u + f(x).$$

This is a steady state problem, otherwise the left hand would have "u_t". It is actually an ODE problem, even though I write it with partial derivative (subscript) notation: $u_{xx} = u''$. The physically-inclined reader should note that the units of the quantities u and f only become clear if one "puts back in" the density, specific heat, and conductivity constants in the equation, as in the conservation of energy derivation of the equation. In that careful view, u is a temperature and f is a heat flux.

(a) Get the following M|O program at `http://www.dms.uaf.edu/~bueler/P16forward.m`, which uses a heat source which is a sum of three concentrated heat sources with shape $e^{-\alpha|x|}$:

```
function u = P16forward(N)
% P16FORWARD   compute and plot parts of forward heat equation problem
% requires circu.m
h = 1/N;   x = (0:h:1-h)';
%  H  is discretization of   -d^2/dx^2 + 3 I   because heat eqn is   0 = u'' - 3 u + f
H = - (1/(h*h)) * circu([-2 1 zeros(1,N-3) 1]') + 3 * eye(N);
c = [5 1 4];
f = c(1) * exp(-20*abs(x-.4)) + c(2) * exp(-40*abs(x-.5)) + c(3) * exp(-20*abs(x-.75));
u = H \ f;
plot(x,f,'o-',x,10*u,'*-')
legend('heat source f(x)','temperature soln, scaled for clarity: 10 * u(x)')
```

Read it, understand it, and run it with $N = 20$, $N = 100$. The standard finite difference approximation $u''(x_j) \approx h^{-2}(U_{j-1} - 2U_j + U_{j+1})$ is used, periodically in $j = 1, 2, \ldots, N$.

Explain briefly why the returned matrix $H_N = H$, from calling the problem with N points in the interval $[0, 1)$, is the discretization of $H_\infty = -d^2/dx^2 + 3$. Justify why the matrix-vector equation solved by the program is the discrete version of $H_\infty u = f$. Explain why we have implemented the periodic boundary values for this ODE correctly.

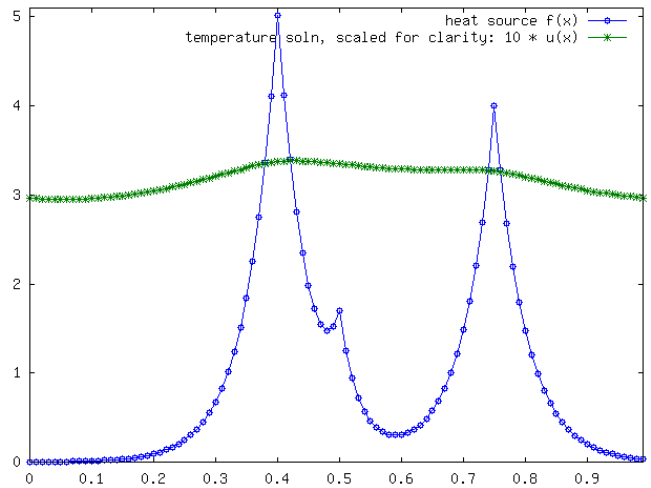

FIGURE 1. Output of `P16forward(100);`

(b) Rewrite the program slightly, with new name `P16map`, for example. It is a M|O function which takes two arguments, `N` and `c`, where `N` is as before and `c` is a real vector of length 3. It solves

$$0 = u_{xx} - 3u + \sum_{j=1}^{3} c_j \phi_j(x) \tag{1}$$

where ϕ_j are the heat source functions already in the program: $\phi_j(x) = \exp(-\alpha_j|x - X_j|)$, $\alpha_j = 20, 40, 20$, $X_j = .4, .5, .75$. For fixed N the new program can be thought of as a function (map)

$$\texttt{P16map}(N, \cdot): \quad c \in \mathbb{R}^3 \mapsto u \in \mathbb{R}^N.$$

Show this map is linear, assuming when needed that the ODE problems you describe have unique solutions.

(c) Because the map is linear, you can: Use your program to produce a $N \times 3$ matrix A which replaces the program. In particular, fix $N = 100$ and run your program a small number of times to compute the columns of $A \in \mathbb{R}^{N\times 3}$ so that we have this equivalence:

$$u = \texttt{P16map}(N, c) \qquad \Longleftrightarrow \qquad u = Ac.$$

Show only the $j = 50$ row of A, not all of A. (*Hint*: Apply to standard basis. Superposition!)

(d) Use the QR "modern classical" method for this least squares problem, an *inverse* problem: Fix $N = 100$. Given $u \in \mathbb{R}^N$, find $c \in \mathbb{R}^3$ so that problem (1) has (discretized) solution u in a least squares sense. Concretely, give c for $u_j = 0.3 + 0.05 \sin(\pi j/N)$, $j = 1, \ldots, N$.