Assignment # 6

Due Monday 30 March at start of class

Exercise 10.1.

Exercise 11.2. **(a)**

Exercise 11.3.

P14. At

`http://www.dms.uaf.edu/~bueler/housemovie.m`

there is a M|O program, demonstrated in class, that shows the steps of the QR decomposition done by Householder reflectors. In particular, it computes R. The information necessary to reconstruct Q is lost. Recall, however, that Algorithms 10.1, 10.2, 10.3 in TREF & BAU compute and use the "v_k" vectors on which the Householder reflectors are based. You can build Q from these if needed, but you frequently don't need to, as in the following.

(a) Do **Exercise 10.2 (a)** in the textbook, possibly based on `housemovie.m`, or starting over. The function should be "clean", and not display anything or ask for input, but just do `[W,R]=house(A)` as stated.

(b) Now modify what you have to make a new procedure `x=housesolve(A,b)`. It takes, as input, a square matrix `A` and a column vector `b`. It computes the solution `x` to the system `Ax`=`b`. Demonstrate that it works by comparing some random systems to the result of `A\b`. (*Use the idea in Algorithm 10.2. Do not form Q. It should be a very short code!*)

(Extra Credit (easy)) Modify to `M=houseinv(A)`, computing `M`=`A`$^{-1}$. Again, do not form Q.

P15. A *circulant matrix* is one where the diagonals are constant and "wrap around":

$$C = \begin{bmatrix} c_1 & c_m & \dots & c_3 & c_2 \\ c_2 & c_1 & c_m & & c_3 \\ \vdots & c_2 & c_1 & \ddots & \vdots \\ c_{m-1} & & \ddots & \ddots & c_m \\ c_m & c_{m-2} & \dots & c_2 & c_1 \end{bmatrix} \tag{1}$$

Said a different way, the entries of C are a function of the difference of the row and column indices, mod m:

$$C_{jk} = \begin{cases} c_{j-k+1}, & j \geq k, \\ c_{m+j-k+1}, & j < k. \end{cases}$$

Here $c_1, \dots, c_m$ can be regarded as the entries of a column vector, the first column of C. In fact we will denote the first column of C by "v". Specifying the first column of a circulant matrix describes it completely.

Get this M|O function, which builds a circulant matrix with a given first column:

`http://www.dms.uaf.edu/~bueler/circu.m`

Notice how it uses the `mod()` function.

(a) Use `circu.m` to generate some simple circulant matrices. In particular, recall that $\{e_j\}_{j=1}^m$ denotes the standard basis for $\mathbb{C}^m$. Let $m = 6$ and use $v = e_1$ to generate circulant matrix C_1 and $v = e_2$ to generate C_2. Note C_1 is just the identity. What is the inverse of C_2? Explain why C_2 is the "downshift" matrix.

(b) Define the *periodic convolution of vectors* $u, w \in \mathbb{C}^m$ by

$$(u * w)_j = \sum_{k=1}^{m} u_{\mu(j,k)} v_k \qquad \text{where} \qquad \mu(j,k) = \begin{cases} j - k + 1, & j \geq k, \\ m + j - k + 1, & j < k. \end{cases}$$

Show, just to exercise this notation, that $u * w = w * u$.

(c) Show that $Cu = v * u$.

(d) Here is an extraordinary fact about circulant matrices: Every circulant matrix has a complete set of eigenvectors that are known in advance, without knowing the eigenvalues.

Specifically, define $f_k \in \mathbb{C}^m$ by

$$(f_k)_j = \exp\left(-i(j-1)(k-1)\frac{2\pi}{m}\right) = e^{-i2\pi(k-1)(j-1)/m},$$

where, as usual, $i = \sqrt{-1}$. Convince yourself that these vectors are *waves*, i.e. combinations of familiar sines and cosines. In particular, choose $m = 100$ and $k = 0, 1, 2, 3, 49$ and clearly plot the real and imaginary parts of f_k.

(e) Now, for the circulant matrix C in (1) above, confirm the "extraordinary fact" as follows: Give a formula for the eigenvalues λ_k, in terms of the entries $c_1, \ldots, c_m$, by showing by direct by-hand calculation that $Cf_k = \lambda_k f_k$.

(Extra Credit 1 (easy and fun)) Before running this, predict what will happen:

```
C=circu(randn(25,1)); [V,D]=eig(C); plot(V), axis equal, axis off
```

Now run it, several times. Explain. Groovy!

(Extra Credit 2) Consider a constant-coefficient linear ODE, of possibly high order, on an interval with periodic boundary conditions. The ODE might be written "$(\mathcal{L}y)(x) = f(x)$" for a constant-coefficient linear differential operator $\mathcal{L}$. Consider applying a finite difference discretization, using an equally-spaced mesh with m spaces. Explain how this gives a circulant matrix problem, and, to the extent possibly, why the ideas in (c) and (e) are not surprising.

At some point it may be useful to see the following Wikipedia pages. No secret answers are revealed, but it may help you organize thoughts:

`http://en.wikipedia.org/wiki/Circulant_matrix`
`http://en.wikipedia.org/wiki/Discrete_Fourier_transform`