Selected Solutions to Assignment # 1

Exercise 1.1. The following is at `http://www.dms.uaf.edu/~bueler/exer1pt1.m` :

```
% exercise 1.1 in Tref&Bau
B = 6*hilb(4)  % make a convenient 4x4 example;
               %   many other ways o.k. but rand() is inconvenient---why?
% part (a)
% MATRIX:                    ACTION ON B:
  M1 = diag([2 1 1 1]);      B*M1
  M2 = diag([1 1 .5 1]);     M2*B*M1
  M3 = [1 0 1 0;
        0 1 0 0;
        0 0 1 0;
        0 0 0 1];            M3*M2*B*M1
  M4 = [0 0 0 1;
        0 1 0 0;
        0 0 1 0;
        1 0 0 0];            M3*M2*B*M1*M4
  M5 = [1 -1  0  0;
        0  1  0  0;
        0 -1  1  0;
        0 -1  0  1];         M5*M3*M2*B*M1*M4
  M6 = [1 0 0 0;
        0 1 0 0;
        0 0 1 1;
        0 0 0 0];            M5*M3*M2*B*M1*M4*M6
  M7 = [0 0 0;
        1 0 0;
        0 1 0;
        0 0 1];              M5*M3*M2*B*M1*M4*M6*M7
% part (b)
A = M5*M3*M2;
C = M1*M4*M6*M7;             A*B*C
% morals drawn from exercise 1.1:
%   1.  All seven operations *can* be written as matrix multiplication.
%   2.  Matrix multiplication is associative so I don't need lots
%       of parentheses (though initially I had them ...).
%   3.  Column ops on the right, row ops on the left.
%   4.  It is nice to both use the command line *and* have a script
%       to do a job of even this modest complexity.
```

P2. (*a*) m^2 multiplications and $m(m-1)$ additions.

(*b*) m versions of the above: m^3 multiplications and $m^2(m-1)$ additions.

(*c*) The number of scalar multiplications in computing an $m \times m$ determinant directly by expansion in minors is m times the multiplications in computing the determinant of an $(m-1) \times (m-1)$ matrix, plus an additional m multiplications (and some additions/subtractions which we are not counting). In fact, let a_m be the number of multiplications to compute the determinant this way on the $m \times m$ case. Note $a_1 = 0$, and $a_2 = 2$, and an easy case to check is $a_3 = 3a_2 + 3 = 9$. Generally,

$$a_m = m\, a_{m-1} + m = m\,(a_m + 1).$$

This is a linear recursion, but a variable coefficient one. The most compact ways I know of writing a_m in "closed" form are

$$a_m = \sum_{k=1}^{m} \frac{m!}{k!} = m! + \sum_{k=2}^{m} \frac{m!}{k!}.$$

The second form is listed so as to emphasize that the work of this algorithm exceed $m!$. *Bad.*

Note there is a really short recursive MOP implementation. See `http://www.dms.uaf.edu/~bueler/baddet.m` This implementation is proof, by example, that short programs can be bad even if formally correct!

P3. *There are many "proofs" in undergraduate linear algebra texts for the parts of theorem 1.3 in* TREF&BAU*. Few of such texts mention singular values, so I will skip part (f) even though it will be a main topic for us in a week or two. Note that the definition of "nonsingular" in* TREF&BAU *is essentially (b) below, but there is no reason to use the word "nonsingular" in the proof below. Clarity suggests you do not use it, though once the theorem is proved it makes sense to use "nonsingular" to mean any of the equivalent conditions! The following proof is slightly more than 10 lines, but it* is *a proof, which was somewhat more than asked.*

Theorem. *For $A \in \mathbb{C}^{m\times m}$, the following conditions are equivalent:*

(a) *A has an inverse A^{-1},*
(b) $\operatorname{rank}(A) = m$,
(c) $\operatorname{range}(A) = \mathbb{C}^m$,
(d) $\operatorname{null}(A) = \{0\}$,
(e) 0 *is not an eigenvalue of A,*
(g) $\det(A) \neq 0$.

Proof. Conditions (b) and (c) are equivalent because the (column) rank is the dimension of range(A) (the column space of A) by definition and also $\mathbb{C}^m$ is the only m dimensional subspace of itself.

If $\operatorname{null}(A) = \{0\}$ then $Av = 0v$ implies $v = 0$, that is, 0 is *not* an eigenvalue. In fact (d), (e) are equivalent.

If $\operatorname{rank}(A) = m$ then Gauss-Jordan elimination by row operations will reduce A to the identity because row operations preserve the span of the columns. Thus (b) implies (a)—the inverse A^{-1} can be built by elimination. If (a) then $Av = 0$ implies $v = A^{-1}0 = 0$, that is, (d). Note row operations do not change the invertibility of $\det(A)$ (i.e. whether $\det(A)$ is nonzero). Thus $\det(A) = 0$ if and only if the end result of elimination has a zero column. Since knowing (d) is equivalent to knowing the columns are linearly independent, (d) implies (g). Finally, if (g) then Cramer's rule allows the inverse to be built, that is, (a). □

The logic of my (outlined) proof is

$$\begin{array}{ccccccc} (c) & \Longleftrightarrow & (b) & \Longrightarrow & (a) & & \\ & & \Uparrow & & \Downarrow & & \\ & & (g) & \Longleftarrow & (d) & \Longleftrightarrow & (e). \end{array}$$

P4. Here is all that is required:

```
>> A = rand(6)
>> B = inv(A)
>> max(max(A)), max(max(abs(B)))  % or you might subtract these ...
>> max(max(abs(A*B - eye(6))))
```

Note that other ways of measuring the difference between `A*B` and `eye(6)` include the matrix norms:

```
>> norm(A*B - eye(6)),    norm(A*B - eye(6),1),    norm(A*B - eye(6),"inf")
```

But this is getting ahead ...