Assignment # 1

Due Friday 30 January at start of class

Exercise 1.1. (page 9, Lecture 1 in TREFETHEN & BAU)

P1. *The purpose of this peculiar problem is to emphasize a necessary mode of thought for applied mathematics. Mathematicians habitually ask, before other questions,* "what is the object?" *and/or* "where does it live?"

(a) Suppose $A \in \mathbb{R}^{4\times 3}$ and $B \in \mathbb{R}^{4\times 5}$ are matrices and $v \in \mathbb{R}^5$ is a column vector. Explain why "ABv" makes no sense, both in terms of matrix manipulation and in terms of the meaning of A, B as linear maps on vector spaces.

(b) Suppose instead that $B \in \mathbb{R}^{2\times 5}$, while A and v have the same size as above. Explain why "ABv" still makes no sense directly but an interpretation can be given, without loss of information in A or B or v, if an additional choice is made. Show the effect of this choice at the level of matrices.

I don't advocate using the expression "ABv" in the (b) case. Rather, my point is that there are different levels of nonsense in applied mathematics, rescue-able and not, depending in part on where things live.

P2. Suppose $A, B \in \mathbb{R}^{m\times m}$ and $v \in \mathbb{R}^m$ is a column vector.

(a) Count all arithmetic operations (additions, subtractions, multiplications, divisions) to compute Av. (*Easy.* No trick.)

(b) Count all arithmetic operations to compute AB.

(c) You learned a method for computing determinants in your linear algebra class, expansion by minors. Count the *multiplication* operations needed to compute $\det(A)$ by this method. (Hint: *How much more work is the* $m \times m$ *case than the* $(m-1)\times(m-1)$ *case?*)

P3. By retrieving or appropriating an undergraduate text on linear algebra, find the proof of Theorem 1.3 (as stated in TREFETHEN & BAU). In roughly 10 lines of text, outline this proof. Comment on undefined terms, relative to the undergrad text ...

P4. *The purpose of this problem is just to get used to playing with the computer.*
Use MATLAB|OCTAVE|PYLAB to do the following operations:

- Build $A \in \mathbb{R}^{6\times 6}$ with random entries. In particular, make the entries uniformly distributed on the interval $(0, 1)$.
- Compute A^{-1} if possible, and call the computed inverse B.
- Compare the largest entry of A to the largest magnitude entry of B.
- How close is AB to the 6×6 identity matrix? Give a MATLAB|OCTAVE|PYLAB expression which computes the maximum amount by which an entry of AB differs from the corresponding entry of the 6×6 identity matrix.

On proving and writing proofs.

On future assignments you will be asked to "show that ..." or "prove that ...". In that case you must clearly understand the range of cases you are addressing. (This is essentially the same as understanding what assumptions you may make.) And you must understand the conclusion you wish to draw. Then you must make an appropriately general, precise, and complete argument which shows why your assumptions imply your conclusion. That is, you need to *prove*. A proof is just a careful argument that goes with the complete logical understanding of a situation. That's math.

I recommend the style of proof below as a template for such an argument. You are not obliged to use the template, but you must still make the careful and complete argument!

Here is **Exercise 1.3** (in Lecture 1 of TREFETHEN & BAU) written out in a good style. (It is a *hard* exercise by the standards of this class, by the way.)

Lemma. *Suppose the columns of a nonsingular upper triangular matrix R are denoted $r_1, \dots, r_m$. For each $j = 1, \dots, m$, it follows that* $\text{span}\{r_1, \dots, r_j\}=\text{span}\{e_1, \dots, e_j\}$. *(Here e_i is the usual $m \times 1$ standard unit column vector.)*

Proof. By induction using contradiction at each induction step. Note $\text{span}\{r_1\} \subseteq \text{span}\{e_1\}$ because R is upper triangular. If "$\subsetneq$" then $r_1 = 0$, but then the column rank of R is less than m, contradicting the nonsingularity of R.

Suppose the statement true for $j = k$ (*induction hypothesis*). Consider $V_{k+1} = \text{span}\{r_1, \dots, r_{k+1}\}$. Note $V_{k+1} \supseteq V_k$. By the induction hypothesis, $V_k = \text{span}\{e_1, \dots, e_k\}$. Because R is upper triangular, if $V_{k+1} \neq \text{span}\{e_1, \dots, e_{k+1}\}$ then $V_{k+1} = \text{span}\{e_1, \dots, e_k\}$. But in that case it follows that the first $k+1$ columns of R can be replaced by $[e_1|\dots|e_k|\mathbf{0}]$ without reducing the span of the columns. That is, the column rank of R is less than m, a contradiction. Thus the statement is true for $j = k+1$ and for all j by induction. □

Exercise 1.3. Suppose R is an $m \times m$ upper triangular matrix with entries r_{ij} so that $r_{ij} = 0$ if $i > j$. Suppose R is nonsingular. Then R^{-1} is also upper triangular.

Proof. Let $Z = R^{-1}$ with entries z_{ij}. By definition of the inverse, the z_{ij} are the unique numbers such that $e_j = \sum_{i=1}^{m} z_{ij} r_i$, for $j = 1, \dots, m$. But $e_j \in \text{span}\{r_1, \dots, r_j\}$ therefore $z_{ij} = 0$ for $i = j+1, \dots, m$. That is, R^{-1} is upper triangular. □

Note that:

- what I assume is clearly stated (including restating parts of the exercise);
- what I intend to prove (the claim) is clearly stated;
- the proof is separated from the claim, and its beginning and end are indicated; and
- because I realized a subidea needed to be clearly stated, I formulated and proved a lemma.

This proof style helps when determining if an argument does or does not show/prove/explain the claim. For instance, if you find you cannot prove the most general statement, but you can prove something which (for instance) has stronger assumptions but the same conclusion, then that situation is clear. And you will get an appropriate amount of credit. As opposed to nothing, which is the appropriate credit for **C**onfusion **R**esembling **A**lmost a **P**roof.