Solutions to Midterm Quiz

Note: *Many of you ran out of time so I chose best 7 out of 8.*
Thus the total possible was 35 points.

1. Definition. Suppose $\|\cdot\|$, $\|\cdot\|'$ are vector norms on $\mathbb{C}^m$, $\mathbb{C}^n$ respectively. Suppose $A \in \mathbb{C}^{m\times n}$ is a matrix. The quantity

$$\|A\| \equiv \max_{x\in\mathbb{C},\, x\neq 0} \frac{\|Ax\|}{\|x\|'}$$

is a matrix norm[1] and is called the *norm induced* by $\|\cdot\|, \|\cdot\|'$.

2. If P is a projector then $\text{range}(I-P) = \text{null}(P)$.

Proof. Suppose $x \in \text{range}(I-P)$ so $x = (I-P)y$. Then $Px = P(I-P)y = (P-P^2)y = 0$ because $P^2 = P$. Thus $x \in \text{null}(P)$. Conversely, if $x \in \text{null}(P)$ the $Px = 0$. That is, $x - Px = x$ so $(I-P)x = x$. It follows that $x \in \text{range}(I-P)$. □

3. If $P^* = P$ then $\text{range}(P) \perp \text{null}(P)$.

[If you use the fact that $\text{null}(P) = \text{range}(I-P)$ at the beginning of the proof, then you will need the hypothesis that P is a projector (i.e. $P^2 = P$). However, the proof below does not need that hypothesis. That is, every Hermitian matrix has range orthogonal to null space, and in fact $\text{range}(A^*) \perp \text{null}(A)$ for any A.]

Proof. If $x \in \text{range}(P)$ then there exists x such that $x = Pz$. If $y \in \text{null}(P)$ then $Py = 0$. But then $x^*y = (Pz)^*y = z^*P^*y = z^*Py = 0$ since $P^* = P$. □

4. If $A = U \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^*$ and $u_j = (-1)^j v_j$ then the eigenvalues of A are $-3, 2, -1$.

Proof. Note $Av_1 = \sigma_1 u_1 = 3u_1 = 3(-v_1) = -3v_1$, $Av_2 = \sigma_2 u_2 = 2u_2 = 2v_2$, $Av_3 = \sigma_3 u_3 = 1u_3 = 1(-v_3) = -v_3$; also note that $v_j \neq 0$. □

[Note: You can also show $A = V\left(\texttt{diag}([-3 \quad 2 \quad -1])\right)V^*$ and the fact that eigenvalues do not change under similarity.]

5. For any matrix A, row rank equals column rank.

Proof. Let $A = U\Sigma V^*$ be an SVD. Suppose $r = (\#$ of nonzero singular values of $A)$. Note $\text{colrank}(A) = \text{colrank}(U\Sigma)$ since V^* is invertible (it is unitary). But

$$\text{colrank}(U\Sigma) = \text{colrank}\left(\left[\begin{array}{c|c|c|c|c|c} \sigma_1 u_1 & \dots & \sigma_r u_r & 0 & \dots & 0 \end{array}\right]\right) = r.$$

On the other hand, $\text{rowrank}(A) = \text{colrank}(A^*) = \text{colrank}(V\Sigma^*)$ because U^* is invertible (and $A^* = V\Sigma^*U^*$). By the same argument as for $U\Sigma$, $\text{colrank}(V\Sigma^*) = r$. □

[1] That is, it satisfies: $\|A\| \geq 0$ and $\|A\| = 0 \iff A = 0$; $\|\lambda A\| = |\lambda|\|A\|$; $\|A+B\| \leq \|A\| + \|B\|$; and $\|AB\| \leq \|A\|\|B\|$. But you need not prove this when stating a definition.

6. Suppose $A = QR$ where $Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$, and suppose $b = \begin{bmatrix} 2\sqrt{2} \\ -\sqrt{2} \\ 3 \end{bmatrix}$. The unique solution to $Ax = b$ is $x = \begin{bmatrix} -4 \\ 6 \\ -1 \end{bmatrix}$.

Proof. Since R is invertible, $A = QR$ is invertible, and there is a unique solution. Consider the following equivalences: $Ax = b \iff Q(Rx) = b \iff Rx = Q^*b$. The last system is easy to solve by hand. First,

$$Q^*b = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} \\ -\sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}.$$

Next we solve a triangular system

$$Rx = Q^*b \quad \iff \quad \begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 5x_3 & = & 3 \\ & & x_2 & + & 3x_3 & = & 3 \\ & & & & -x_3 & = & 1 \end{array}$$

by back substitution: $x_3 = -1$, $x_2 = 3 - 3(-1) = 6$, and $x_1 = 3 - 2(6) - 5(-1) = -4$. □

7. Count operations in Algorithm 7.1, Classical Gram-Schmidt.

Solution. First, lines $\boxed{3}$ and $\boxed{4}$ are executed only n times. (Specifically, they require $n \cdot (1 + m + (m-1) + m) = 3mn$ flops total.) Thus we ignore them. The remaining work is $2mn$, that is, $K = 2$:

$$\begin{aligned} \sum_{j=1}^{n} \sum_{i=1}^{j-1} (m + (m-1)) + (m+m) &= \sum_{j=1}^{n} \sum_{i=1}^{j-1} 4m - 1 \sim \sum_{j=1}^{n} \sum_{i=1}^{j-1} 4m \\ &= 4m \sum_{j=1}^{n} (\overbrace{1 + \cdots + 1}^{j-1 \text{ ones}}) = 4m \sum_{j=1}^{n} j - 1 = 4m \sum_{k=1}^{n-1} k \\ &= 4m \, \frac{(n-1)n}{2} \sim 2\,mn. \end{aligned}$$

□

8. Problem. Given $A \in \mathbb{C}^{m \times n}$, find a rank one matrix $\hat{B} \in \mathbb{C}^{m \times n}$ that minimizes the function

$$f(B) = \|A - B\|_2$$

over all rank one matrices $B \in \mathbb{C}^{m \times n}$.

Solution. $\hat{B} = \sigma_1 u_1 v_1^*$, where $A = U\Sigma V^*$ is any SVD of A. □