Final Exam

90 points total—**F5** made Extra Credit!

Due *Wednesday December 17, 2003 at 12:15 pm*
in my office or mailbox.

Rules. You may not talk or communicate with any person *other than me* (Ed Bueler), about this exam. You may use any reference, print or electronic, as long as it is clearly cited. You may even want to add a references section to your solutions! References to the textbook may be omitted, in general, though you may add them for clarity.

F1. (*5 pts for each part*). Consider Gaussian elimination for square tridiagonal matrices, and also consider algorithms for Hermitian positive definite tridiagonal matrices. In particular:

Definition. A matrix $A \in \mathbb{C}^{m\times m}$ is *tridiagonal* if $a_{ij} = 0$ for all i, j such that $|i - j| > 1$.

(a) State and prove a theorem for the *exact* amount of work done in solving a tridiagonal, $m \times m$ system $Ax = b$ by Gaussian elimination *without* pivoting.

(b) State and prove a theorem which quantitatively describes how the algorithm LU DECOMPOSITION BY GAUSSIAN ELIMINATION WITH PARTIAL PIVOTING, that is, computation of $PA = LU$, will destroy the tridiagonalness of A. That is, quantify how much partial pivoting destroys the tridiagonal structure in the worst case. Consider, in particular, how many nonzero entries can appear in U, in the worst case.

(c) TREFETHEN & BAU Exercise **25.1** (a)

(d) Write a working MATLAB `m`-file for tridiagonal Cholesky decomposition $A = R^*R$ (of a Hermitian tridiagonal positive definite matrix A). State exactly how much work this algorithm requires as a function of m.

(e) Produce a 100×100 Hermitian positive definite tridiagonal matrix A with random *off*-diagonal elements directly produced from `randn`, in MATLAB. Note that roughly half of the off-diagonal elements will be negative. *Do not show me the entries; show me how you created the matrix!* Demonstrate clearly that your matrix is in fact positive definite. Use the result of **(d)** to solve a system of equations with this matrix A and compare the result to MATLAB's "`A\b`". *Do not show me the entries of* x. Instead, clearly indicate the size of the difference between your result and MATLAB's result.

(f) Use the power method, at the command line in MATLAB, to find the largest eigenvalue of A in part **(e)**. Provide evidence that your estimate is accurate to at least 4 digits.

F2. (*5 pts for each part*). (*The result of this problem should be a clear mental picture of underdetermined systems, and a bit of their numerical analysis.*)

Consider an $m \times n$ matrix A where $m < n$ and A has full rank.

(a) Carefully and mathematically explain why the system $Ax = b$ for this matrix is an *underdetermined*.

Because the system is underdetermined and has many solutions one must give further properties which determine a single desired solution. In this case, we decide that the desired solution is $\tilde{x}$ where $A\tilde{x} = b$ and $\|\tilde{x}\|_2 = \min\{\|x\|_2 \big| Ax = b\}$. (You might compare this situation to that of overdetermined systems (Lecture 11).)

(b) Recalling Figure 11.3 of the text, draw the analogous picture for an underdetermined system. (*Hint*: Your picture should be a picture of $\mathbb{C}^n$, the input space of A, rather than $\mathbb{C}^m$ as in Figure 11.3. Include $\text{null}(A)$ and $\text{range}(A^*)$ in your picture.)

(c) Use the SVD of A to describe an algorithm for finding $\tilde{x}$.

F3. (*15 pts*). Choose one of the following:

Trefethen & Bau Exercise **22.1**.

Trefethen & Bau Exercise **22.3**.

F4. (*15 pts*). Trefethen & Bau Exercise **24.4**.

F5. (*3 pts Extra Credit*). This one is kind of silly: Trefethen & Bau Exercise **4.2**.

Alternate or Extra Credit. (*variable pts*). You may substitute the following problem for any one of the other problems on this exam (i.e., **F1**, ..., **F6**) *or* you may do it for extra credit which will be worth *10 pts*:

Trefethen & Bau Exercise **26.1**.

F6. (*15 pts*). Fill in the third and fourth columns of the following table as they apply to $m \times n$ matrices. You may or may not want to recopy the table on your own paper. Follow the rough model of the places I have filled in. If you cannot determine a block, please write "UNKNOWN". (For instance, I have put that in one entry below; you need not address that entry further.)

Note. If you put "UNKNOWN" in blocks where you *should* know, you lose points!

For the "Work" column, if possible write the leading order work as a function (typically power) of m with the correct constant, as in the example. If that is not possible, write "$O(f(m))$" as appropriate.

For the "Stability" column, write "unstable", "stable", or "backward stable," as appropriate. If further comment is necessary, add a footnote.

Problem	**Algorithm**	**Work**	**Stability**
$Ax = b$, $m < n$, A full rank, $\|x\|_2$ minimum	SVD		UNKNOWN
$Ax = b$, $m > n$, A full rank, $\|r\|_2 = \|b - Ax\|_2$ minimum	normal eqns and Cholesky		
"	QR		
"	SVD		
$Ax = b$, $m = n$, A nonsingular	Gauss elim. (and back sub.) w/o pivoting	$\mathbf{\frac{2}{3}m^3 + O(m^2)}$	**unstable**
"	Gauss elim. w partial pivoting		
"	Gauss elim. w complete pivoting		
"	QR and back sub.		
"	SVD		
$Ax = b$, A Herm pos def	Cholesky		
$Rx = b$, $m = n$, R triangular	back substitution		
$Tx = b$, $m = n$, T tridiagonal	Gauss elim. w/o pivoting		
$Ux = b$, U unitary	matrix multiplication		
$\det(A)$, $m = n$	expansion in minors (by hand algorithm)		
$\det(A)$, $m = n$	Gauss elim. w partial pivoting		
$\lvert\det(A)\rvert$, $m = n$	QR		