Assignment # 6

Due *Wednesday 11/5* at start of class

Exercise 10.1.

Exercise 11.2. (a)

Exercise 11.3.

IX. What good are the orthogonal functions $q_j(x)$, $j = 1, \dots, 5$, which were the columns of Q in problem **VIII**? One answer is that they can sometimes help to approximately solve hard problems which relate in some manner to the functions e^{-jx}. For instance:

Problem. *Find a smooth function $\hat{u}(x)$, defined for $x \in [0, \infty)$, which approximately solves*

$$-u''(x) + (5 + 4\cos(x))\, u(x) = 0, \quad u(0) = 1, \quad \lim_{x \to +\infty} u(x) = 0.$$

Of course, the sense of "approximation" is vague, and I will leave it so. (Try to come up with a better approximation than the one given, using only four coefficients! By the way, I do *not* know how to solve this problem by hand. Do you?)

(a) Argue informally that the exact $u(x)$ satisfies $e^{-3x} \le u(x) \le e^{-x}$. (*Hint:* Compare the problem to one with "$5 + 4\cos(x)$" replaced by constants. Solve those problems exactly.)

Now consider (and run!) the following MATLAB :

```
>> B=hilb(6); B=B(1:5,2:6); R=chol(B);
>> x=(0:.01:3)'; A=[exp(-x) exp(-2*x) exp(-3*x) exp(-4*x) exp(-5*x)];
>> Q=A/R; plot(x,Q(:,5)), grid on
>> c=roots(flipud(R\([zeros(1,4) 1]'))); z=-log(c); z'
 ans =
     0.0588    0.3241    0.8761    1.9678
```

(b) Explain why z contains the four finite solutions of $q_5(x) = 0$.

Suppose $\hat{u}(x)$ is a linear combination of $q_1(x), \dots, q_4(x)$: $\hat{u}(x) = \sum_{j=1}^{4} c_j q_j(x)$. A *spectral collocation method* for solving the problem above is to use the values $z_1, \dots, z_4$, and the boundary condition at $x = 0$, to determine c_j (and thus $\hat{u}(x)$) by the following prescription:

$$\sum_{j=1}^{4} c_j q_j(0) = 1, \tag{1}$$

$$\sum_{j=1}^{4} c_j \left(-q_j''(z_k) + (5 + 4\cos(z_k)) q_j(z_k) \right) = 0, \quad k = 1, 2, 3, 4. \tag{2}$$

This is five equations, the first of which is the boundary condition, and the next four are requiring the differential equation to be true at the roots of $q_5(x)$. As there are only four unknown coefficients, the system is overdetermined.

(c) Using either a least squares method to approximately solve equations (1) and (2), *or* removing the z_4 equation from equations (2) and solving the resulting system, find c_j and plot the solution $\hat{u}(x)$.

Extra Credit:

X (Extra Credit). Now find some other way to approximately solve the above boundary value problem, presumably using a method (finite differences?) with lots of degrees of freedom, and compare. Explain what you do with the boundary at infinity in this context.