Assignment # 1

Due Wednesday 9/10 at start of class

I. By retrieving or appropriating an undergraduate text on linear algebra, find the proof of Theorem 1.3 (as stated in Trefethen). In roughly 10 lines of text, outline this proof. (Comment on undefined terms, relative to the undergrad text . . .)

Exercise 1.1. (page 9, in Lecture 1 in Trefethen)

Exercise 1.4. (page 10)

On proving and writing proofs.

If you are asked to "show that …" or "prove that …" then you must clearly understand the range of cases you are addressing. This is essentially the same as understanding what assumptions you may make, of course. Also, you must understand the conclusion you wish to draw. Then you must make an appropriately general, precise, and complete argument. That is, you need to *prove*. On the other hand, a proof is just a careful argument that goes with the complete logical understanding of a situation. That's math.

So I recommend the style of proof below as a template for such a careful and complete argument. You are not obliged to use the template, but you must still make the careful and complete argument!

Here is **Exercise # 1.3** (in Lecture 1 of Trefethen) written out in a good style. (It is a *reasonably hard* exercise by the standards of this class, by the way.)

Lemma. *Suppose the columns of a nonsingular upper triangular matrix R are denoted $r_1, \dots, r_m$. For each $j = 1, \dots, m$, it follows that* $\text{span}\{r_1, \dots, r_j\} = \text{span}\{e_1, \dots, e_j\}$. *(Here e_i is the usual $m \times 1$ standard unit column vector.)*

Proof. By induction using contradiction at each induction step. Note $\text{span}\{r_1\} \subseteq \text{span}\{e_1\}$ because R is upper triangular. If "$\subsetneq$" then $r_1 = 0$, but then the column rank of R is less than m, contradicting the nonsingularity of R.

Suppose the statement true for $j = k$ (*induction hypothesis*). Consider $V_{k+1} = \text{span}\{r_1, \dots, r_{k+1}\}$. Note $V_{k+1} \supseteq V_k$. By the induction hypothesis, $V_k = \text{span}\{e_1, \dots, e_k\}$. Because R is upper triangular, if $V_{k+1} \neq \text{span}\{e_1, \dots, e_{k+1}\}$ then $V_{k+1} = \text{span}\{e_1, \dots, e_k\}$. But in that case it follows that the first $k+1$ columns of R can be replaced by $[e_1|\dots|e_k|\mathbf{0}]$ without reducing the span of the columns. That is, the column rank of R is less than m, a contradiction. Thus the statement is true for $j = k+1$ and for all j by induction. $\square$

Exercise 1.3. Suppose R is an $m \times m$ upper triangular matrix with entries r_{ij} so that $r_{ij} = 0$ if $i > j$. Suppose R is nonsingular. Then R^{-1} is also upper triangular.

Proof. Let $Z = R^{-1}$ with entries z_{ij}. By definition of the inverse, the z_{ij} are the unique numbers such that $e_j = \sum_{i=1}^m z_{ij} r_i$, for $j = 1, \dots, m$. But $e_j \in \text{span}\{r_1, \dots, r_j\}$ therefore $z_{ij} = 0$ for $i = j+1, \dots, m$. That is, R^{-1} is upper triangular. $\square$

Note that:

- what I assume is clearly stated (including restating parts of the exercise);
- what I intend to prove (the claim) is clearly stated;
- the proof is separated from the claim, and its beginning and end are indicated; and
- because I realized a subidea needed to be clearly stated, I formulated and proved a lemma.

This proof style helps when determining if an argument does or does not show/prove/explain the claim. For instance, if you find you cannot prove the most general statement, but you can prove something which (for instance) has stronger assumptions but the same conclusion, then that situation is clear. And you will get an appropriate amount of credit. As opposed to nil, the appropriate credit for **C**onfusion **R**esembling **A**lmost a **P**roof.