Assignment #9

Due *Friday May 11, 2007.*

0. Read sections 5.1, 5.2, 5.3, 5.4, 5.5, 6.1, 6.2, 6.3.

1. (*15 points*) Exercise 2.10 in MORTON & MAYERS 2ND ED (page 60).

2. (*15 points*) Exercise 4.4 in MORTON & MAYERS 2ND ED (pages 147–148).

3. (*20 points*) *Implement* method (6.29b) for equation (6.20) in MORTON & MAYERS 2ND ED. In particular, approximate the solution $u(x, y)$ on the square $(x, y) \in [-1, 1] \times [-1, 1]$ if

$$a(x, y) = 0.7 + \frac{x - y}{1 + (x - y)^2}$$

and

$$f(x, y) = \begin{cases} 1, & (x + 0.5)^2 + (y + 0.5)^2 < 0.1, \\ 0, & \text{otherwise.} \end{cases}$$

Use boundary conditions $u = 0$ on all edges of the square. Be sure to use sparse matrix storage. I suggest you just solve the linear system by "`A\b`". *Show* the result for the $\Delta x = \Delta y = 0.1$ and $\Delta x = \Delta y = 0.01$ cases. Note the sizes of these two matrix problems. *Explain* the result by physical reasoning. In particular, interpret $u(x, y)$ as the equilibrium temperature of a square plate with nonconstant conductivity $a(x, y)$ and heat source distribution $f(x, y)$; plotting $a(x, y)$ and $f(x, y)$ will be very useful in giving this explanation.

4. (*20 points*) Read the one page derivation of the telegraph equation by J. Feldman. To give context, and for entertainment, read the several page historical reading from T. Körner's *Fourier Analysis*.

(a) Consider equation (3) in the Feldman notes, called the *telegraph equation*. First, *describe* in a couple of sentences how the telegraph equation is a generalization of the wave equation. In particular, justify physically, in terms of the derivation given by Feldman, which uses a circuit describing a segment of telegraph wire, why the $\alpha = \beta = 0$ case of equation (3) is the wave equation.

(b) Next, download the short MATLAB code

`http://www.dms.uaf.edu/~bueler/waveleap.m`

It solves the wave equation by the leap frog method described in MORTON & MAYERS 2ND ED (section 4.9, pages 125–127). In particular, `waveleap.m` approximates the function $u(x,t)$ which solves the PDE $u_{tt} = c^2 u_{xx}$ in the domain $(x,t) \in [0,100] \times [0,40]$ using speed $c = 1$ and boundary/initial conditions

$$u(0,t) = u(100,t) = 0, \qquad u(x,0) = \exp\left(-(x-50)^2/2\right), \qquad u_t(x,0) = 0.$$

Become familiar with this program and evaluate its accuracy by

i) *adding a comment to each line of the program which explains what it does*; call your new program `waveleapXX.m` where "`XX`" are your initials;

ii) *finding the exact solution of this wave problem using the d'Alembert solution*; note I showed the d'Alembert solution in class but you should be able to look it up in nearly any undergraduate PDE text;

iii) *adding the exact solution to* `waveleapXX.m` *and computing the error at* $t = 40$*, by comparing to the exact solution at* $J = 40, 100, 400, 2000$; *compute and report a convergence rate.*

(c) Now go back to the telegraph equation (3) and apply the leap frog method to it. That is, implement the leap frog method on (3) by *modifying* `waveleapXX.m` to take any α, β. Call your new program `teleleapXX.m`. You will need to derive a new first order pair of equations which generalize (4.98) in MORTON & MAYERS 2ND ED; note that the equation $u_t = v$ must also be solved, as in `waveleap.m`. Your program should still be able to solve the problem in **(a)**, so you may at least check it against the exact solution there. *Show* what your program computes in the $J = 100$ and $J = 1000$ cases if $\alpha = 0.03$ and $\beta = 0.001$. In this case there is a substantial leakage current but the wire is still a good conductor; use the same domain, speed, and boundary/initial conditions as in **(a)**. Note you will need to be careful if you want to preserve the accuracy present in `waveleap.m`. Also, *display* the solutions $u(x, t{=}40)$, for the $\alpha = \beta = 0$ case and for the just-described case with leakage; display as functions of x in the usual way.