Assignment #5

Due *Monday March 5, 2007.*

1. Read sections 2.13, 2.14, 2.15, 2.16, and 2.17 of MORTON & MAYERS, 2ND ED.

2. Exercise 2.6 (page 57).

3. Show that the explicit scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{(U_{j+1}^n - U_j^n)p_{j+1/2} - (U_j^n - U_{j-1}^n)p_{j-1/2}}{\Delta x^2}$$

for the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right)$$

is consistent assuming $p(x)$ has continuous derivative. Note $x_{j+1/2} = (x_j + x_{j+1})/2$ and $p_{j+1/2} = p(x_{j+1/2})$. *This easier exercise replaces Exercise 2.7, page 59.*

Hint: You seek the leading terms in the truncation error. Use Taylor's theorem to get

$$\begin{aligned} &p(x+\epsilon)\left[u(x+\Delta,t) - u(x,t)\right] \\ &\quad = \left(p(x) + p'(\xi)\epsilon\right)\left[u_x(x,t)\Delta + \frac{1}{2}u_{xx}(x,t)\Delta^2 + \frac{1}{6}u_{xxx}(x,t)\Delta^3 + \frac{1}{24}u_{xxxx}(\nu,t)\Delta^4\right]. \end{aligned}$$

Now use this to compute

$$p(x+\Delta x/2)\left[u(x+\Delta x,t) - u(x,t)\right] - p(x-\Delta x/2)\left[u(x,t) - u(x-\Delta x,t)\right],$$

and consider $\lim_{\Delta x\to 0}$ of this quantity divided by Δx^2.

4. **a**. Implement, in MATLAB presumably, the Crank-Nicolson method for the PDE problem

$$u_t = (1+2x)u_{xx} + 3u + f(x,t), \qquad u_x(0) = 0, \quad u(1) = 0. \tag{1}$$

Use method (2.114) (page 42) to implement the boundary condition. Let $\nu = \Delta t/\Delta x$, and suppose we fix $\nu = 1/2$ in both parts **a** and **c** of this problem. Find the numerical solution $u(x, t_f)$ when $f(x,t) = 0$, $u(x,0) = 1 - x$, and $t_f = 1$. Use both $J = 20$ and $J = 100$.

b. Now suppose

$$f(x,t) = \cos\left(\frac{\pi x}{2}\right)\left\{1 + t\left(\frac{\pi^2}{2^2}(1+2x) - 3\right)\right\}.$$

Check by hand that $u(x,t) = t\cos(\pi x/2)$ is an exact solution to PDE boundary value problem (1) with initial value $u(x,0) = 0$. How did I come up with this exact solution?

c. Using the exact solution $u(x,t)$ in **b**, and the particular $f(x,t)$ given there, evaluate the maximum numerical error, using the program you wrote in part **a**, at time $t_f = 1$ and using each of $J = 20, 40, 80, 160, 320$. Display the error versus J in the way(s) you think communicates the important ideas. Discuss what you have (and have not) demonstrated.