Assignment #4

Due *Friday, 23 February, 2007.*

1. Read sections 2.9, 2.10, 2.11, and 2.12 of Morton & Mayers, 2nd ed.

2. Compute the truncation error

$$T(x,t) := \frac{u(x,t+\Delta t) - u(x,t-\Delta t)}{2\Delta t} - \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2}$$

of method (2.98) on page 38. In particular, show that it satisfies

$$T(x,t) = \frac{1}{3} u_{ttt}(x,\tau)\Delta t^2 - \frac{1}{12} u_{xxxx}(\xi,t)\Delta x^2$$

for some $x - \Delta x \le \xi \le x + \Delta x$ and $t - \Delta t \le \tau \le t + \Delta t$. You will use the fact that $u_t - u_{xx} = 0$; recall $u(x,t)$ is the exact solution. Under the hypothesis that u_{tt} and u_{xxxx} are bounded, we see that $T(x,t) = O(\Delta t^2, \Delta x^2)$.

The stability of the explicit three-level scheme (2.98) for the heat equation is already addressed in section 2.12 of the textbook. The method is unconditionally unstable. Thus this method is both significantly more accurate than the simplest explicit method and completely useless. [*It is the method I called the "Richardson" scheme in class; see the handout from the book* The Pleasures of Counting *by Körner.*]

3. Consider applying the implicit method to a heat equation with constant conduction $\kappa > 0$ and "reaction" term with constant rate $C \in \mathbb{R}$: $u_t = \kappa\, u_{xx} + C\, u$. There are, at least, these two implicit schemes:

Scheme I: $$\frac{\Delta_{+t} U_j^n}{\Delta t} = \kappa\, \frac{\delta_x^2 U_j^{n+1}}{\Delta x^2} + C\, U_j^n$$

Scheme II: $$\frac{\Delta_{+t} U_j^n}{\Delta t} = \kappa\, \frac{\delta_x^2 U_j^{n+1}}{\Delta x^2} + C\, U_j^{n+1}$$

Apply the Fourier analysis of section 2.7 to each of these schemes and discuss the result. Does growth always imply instability? Will these schemes be very different in their stability behavior?

4. Compare, in an actual computation, the explicit method (2.19) and the explicit method described in exercise 2.3 (page 58). In particular, use a uniform mesh with $\Delta x = 0.02$ for (2.19) and use the mesh

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x = [0.0:0.01:0.40  0.45:0.05:1.0]
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(Matlab notation) for the exercise 2.3 method. Use initial condition

$$u(x,0) = u^0(x) = \begin{cases} 10\sin(33\pi x), & 0 < x \le 1/3, \\ \sin(3\pi x), & 1/3 < x < 1 \end{cases}$$

and compute approximations of $u(x,t_f)$ for $t_f = 0.02$ by each method. Use the correct Δt which satisfies the stability condition. Discuss. Recall the technical issues which arose in Exercise 2.3 in your discussion. (*Extra credit for comparing to the exact solution.*)