Definitions and Examples

I like our textbook Morton & Mayers. And I think it is not a bad thing to have to say: "I need to augment it sometimes to improve understanding by students." In particular, on the first pass through this material it is a good idea to have very clear definitions for the few most important concepts. These definitions do appear in the text, but here I have pulled them out and put them on display. I have noted the page(s) in Morton & Mayers on which each definition appears.

Definition 1 (Page 13; Page 136). *The* truncation error *(or* local truncation error*) of a numerical scheme at a point (x, t) in the domain is the amount by which the exact solution does not satisfy the scheme at that point. In particular, if the PDE satisfied by the exact solution u is written $F(u) = 0$, and if $\tilde{F}(U) = 0$ is the equation satisfied by the discrete approximation U then the truncation error is $T = \tilde{F}(u)$.*

Example. We can write the standard heat equation as

$$u_t - u_{xx} = 0$$

and the explicit scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} - \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} = 0. \tag{1}$$

Then the truncation error is

$$T(x,t) = \frac{u(x, t + \Delta t) - u(x,t)}{\Delta t} - \frac{u(x + \Delta x, t) - 2u(x,t) + u(x - \Delta x, t)}{\Delta x^2}.$$

Again, the truncation error is a function of the point in the domain; it essentially always varies over the domain.

Definition 2 (Page 16). *A* refinement path *is a sequence of positive mesh parameters $\{(\Delta x_i, \Delta t_i)\}$, $i = 1, 2, \dots$, such that $\Delta x_i \to 0$ and $\Delta t_i \to 0$ as $i \to \infty$.*

Example. A refinement path for the explicit scheme (1) is

$$\{(\Delta x_i, \Delta t_i) = (0.2^i, 0.01^i)\}, \quad i = 1, 2, \dots$$

This happens to be a *stable* refinement path because $\Delta t / \Delta x^2 = (1/4)^i \le 1/2$. But the concept of stability does not enter into the definition of "refinement path." For instance, $\{(\Delta x_i, \Delta t_i) = (0.1^i, 0.1^i)\}$ is also a refinement path, but the explicit method would show instability along it.

Definition 3 (Page 15; Page 137). *A numerical scheme is* consistent (along a refinement path) *if for any point (x,t) in the domain the truncation error goes to zero as the mesh sizes go to zero along the refinement path. A numerical scheme is* unconditionally consistent *if the truncation error goes to zero as the mesh sizes go to zero along* any *refinement path.*

Example. The truncation error of the explicit scheme (1) for the heat equation turns out to satisfy

$$|T(x,t)| \le \frac{1}{2} M_{tt} \Delta t + \frac{1}{12} M_{xxxx} \Delta x^2,$$

where M_{tt}, M_{xxxx} are bounds on derivatives of the exact solution. This is inequality (2.36) in the text; we learn this inequality from Taylor's theorem, of course. Thus the explicit scheme is unconditionally consistent. For instance, the truncation error goes to zero along either of the paths mentioned in the previous example.

Example. Exercise 5.1 on page 159 describes an explicit three level scheme for the equation $u_t + a u_x = b u_{xx}$, called the *DuFort-Frankel scheme*, which is conditionally consistent. That is, one has consistency only along refinement paths which satisfy a condition.

Definition 4 (Page 15)**.** *We say a numerical scheme has* order (of accuracy) *"$O(\Delta t^p + \Delta x^q)$" if there exist $C_1, C_2 > 0$ so that*

$$|T(x,t)| \le C_1 \Delta t^p + C_2 \Delta x^q$$

for all x, t in the domain. (If the order on the spatial mesh is understood to be fixed, then we can sometimes say "pth order accuracy" to denote the power on Δt.)

Example. As mentioned above, the explicit method for the heat equation has order $O(\Delta t^1 + \Delta x^2)$. The Crank-Nicolson scheme has order $O(\Delta t^2 + \Delta x^2)$ (section 2.10). Because the spatial derivative approximations are the same in these two schemes it makes sense, in context, to say that the explicit scheme is "first-order accurate" and that Crank-Nicolson is "second-order accurate."

Definition 5 (Pages 19–20; more general definition given on Page 137)**.** *Suppose we have a constant-coefficient finite difference scheme $\tilde{F}(U) = 0$, with approximate solution $U = (U_j^n)$, for a time-dependent PDE and suppose $U_j^n = \lambda(k)^n e^{ik(j\Delta x)}$ is a spatial frequency k solution to the finite difference scheme. The finite difference scheme is* stable (along a refinement path)*, or* conditionally stable*, if there exists $K > 0$ independent of the $k, \Delta t, \Delta x$ such that*

$$|\lambda(k)^n| \le K \qquad \text{for all frequencies } k \text{ and for all } n \text{ such that } n\Delta t \le t_f$$

for all $(\Delta x, \Delta t)$ on the refinement path. A finite difference scheme is unconditionally stable *if the above condition applies for every refinement path.*

Lemma 6 (von Neumann; Page 20)**.** *If there exists $K' > 0$ such that $|\lambda(k)| \le 1 + K'\Delta t$ for all k and all $(\Delta x, \Delta t)$ on the refinement path then the scheme is stable. In particular, if $|\lambda(k)| \le 1$ for all k and all $(\Delta x, \Delta t)$ on the refinement path then the scheme is stable.*

Example. For the explicit scheme we calculate (Page 19) that

$$\lambda(k) = 1 - 4\nu \sin^2((1/2)k\Delta x),$$

where $\nu = \Delta t/(\Delta x)^2$, and thus this method is stable along any refinement path for which $\nu \le 1/2$. For the implicit scheme we calculate (Page 25) that

$$\lambda(k) = (1 + 4\nu \sin^2((1/2)k\Delta x))^{-1}.$$

Thus the implicit scheme is unconditionally stable.

Example. Let's apply the explicit scheme to a heat-equation-with-linear-sink:

$$u_t = u_{xx} - u \qquad \rightarrow \qquad \frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} - U_j^n.$$

If $U_j^n = \lambda(k)^n e^{ik(j\Delta x)}$ is a solution then

$$\lambda(k) = 1 - 4\nu \sin^2((1/2)k\Delta x) \; - \; \Delta t.$$

If $\nu = 1/2$ and $k_0 = J\pi$ then $\lambda(k_0) = -1 - \Delta t$ so we have $|\lambda(k_0)| > 1$. However, there is $K > 0$ so that if $n\Delta t \leq t_f$ then $|\lambda(k)^n| \leq K$ for all k (including $k = k_0$). This example illustrates the first sentence of von Neumann's lemma. You can check that $K = e^{t_f}$ works because $(1 + \Delta t)^N = (1 + t_f/N)^N \leq e^{t_f}$ (where $N = t_f/\Delta t \approx \max\{n \,|\, n\Delta t \leq t_f\}$). Actual use of a refinement path with $\nu = 1/2$ does not generate instability.

Definition 7 (Page 15)**.** *For a fixed grid on a domain, the* error *(or* actual error*) of a numerical scheme at a grid point* (x_j, t_n) *is the difference*

$$e_j^n = U_j^n - u(x_j, t_n)$$

where $u(x,t)$ *is the exact solution and* U_j^n *is the computed approximation at that grid point. The maximum of* $|e_j^n|$ *over all grid points in the domain is the* global error *for that scheme using that grid.*

Remark. One can also define the *maximum error at each time step*

$$E^n = \max_j |e_j^n|,$$

but this quantity is tends to be just an intermediate quantity which arises in the middle of a proof of convergence. See section 2.6, for example.

Definition 8 (Page 15)**.** *A numerical scheme is* convergent (along a refinement path) *if, for any fixed point* (x^*, t^*) *in the domain,*

$$\Delta x_i \to 0, \; \Delta t_i \to 0, \; x_j \to x^*, \; \textit{and } t_n \to t^* \qquad \textit{implies} \qquad U_j^n \to u(x^*, t^*)$$

for $(\Delta x_i, \Delta t_i)$ *on the refinement path. As long as the exact solution is continuous it suffices to show that that the global error goes to zero along the refinement path (though this condition actually corresponds to a kind of* uniform *convergence).*

Example. On Pages 15–16 it is shown that *if* $\nu \leq 1/2$ then the global error for the explicit method for the heat equation satisfies

$$\max_{(x_j,t_n)\in \text{ grid}} |e_j^n| \leq \frac{1}{2}\Delta t \left[M_{tt} + \frac{1}{6\nu} M_{xxxx} \right] t_f,$$

where M_{tt}, M_{xxxx} are bounds on derivatives of the exact solution. Assuming these bounds exist and assuming we have a refinement path for which $\nu \leq 1/2$, it follows that the explicit method converges.