

Assignment #6

Due *Monday March 21, 2005*.

1. Read sections 2.17, 3.1, 4.1, 4.2.
2. Analyze the θ -method, with $0 \leq \theta \leq 1$ as usual, for

$$u_t = a_0 u_{xx} + b_0 u_x,$$

where a_0, b_0 are constant. That is, derive the equation satisfied by the error $e_j^n := U_j^n - u(x_j, t_n)$, the analog of equation (2.142). [*Hint*: You can check your answer by seeing if it agrees with (2.142) when $\theta = 0$, $a(x, t) = a_0$, $b(x, t) = b_0$, $c(x, t) = 0$, and $d(x, t) = 0$.]

Then derive the sufficient conditions for convergence analogous to (2.144) and (2.145). [*Hint*: There will be two such conditions. One condition will be inactive when $\theta = 1$.]

Conclude by answering the question: Does implicitness help with the (2.144)-type condition which relates the magnitudes of the advection versus conduction coefficients?

3. Apply an adaptive time-stepping explicit method to the following nonlinear heat problem:

$$u_t = (e^{-30 \arctan u}) u_{xx} - 1, \quad u(0, t) = 0, \quad u(1, t) = 0, \quad u(x, 0) = x(1 - x).$$

That is, incorporate the condition (2.171) into your program to choose each time step. Approximate $u(x, t)$ at $t_f = 0.5$. [*Hint*: `exp` and `atan` in MATLAB.]

Comment on how the behavior of the solution affects the time steps. In fact, plot the time steps as a function of the time.

[*Note*: I found that $\Delta x = 0.01$ is a mesh size which produces a reasonable execution time. Nonetheless you should try other values so that you are confident you are seeing a close approximation of the solution. Truthfully, I have no idea what the formula for the exact solution might be. I am, however, completely confident there is only one such solution and that it is smooth.]

4. Implement the Crank-Nicolson method on the heat problem

$$u_t = u_{xx}, \quad u_x(0, t) = 3 \cos(3) e^{-9t}, \quad u(1, t) = 0, \quad u(x, 0) = \sin(3(x - 1)),$$

where $0 \leq t \leq 0.1$ (i.e. $t_f = 0.1$) and $0 \leq x \leq 1$. This problem includes an example of the boundary condition discussed in section 2.13, with $\alpha(t) = 0$ and $g(t) = 3 \cos(3) e^{-9t}$. The exact solution to this mildly-contrived problem is $u(x, t) = e^{-9t} \sin(3(x - 1))$.

Try both numerical boundary conditions (2.103) and (2.114) and compare the error which results. In particular, use a grid with $\nu = 5$ and $J = 100$ intervals.