

Assignment #4

Due *Friday, 25 February, 2005.*

1. Read sections 2.10, 2.11, and 2.12.
2. Compare, in an actual computation, the explicit method (2.19) and the explicit method described in exercise 2.3 (pages 55-56).

In particular, use a uniform mesh with $\Delta x = 0.02$ for (2.19) and use the mesh

$$\mathbf{x} = [0.0:0.01:0.40 \quad 0.45:0.05:1.0]$$

(MATLAB notation) for the method described in exercise 2.3. Use initial condition

$$u(x, 0) = u^0(x) = \begin{cases} 10 \sin(33\pi x), & 0 < x \leq 1/3, \\ \sin(3\pi x), & 1/3 < x < 1 \end{cases}$$

and compute approximations of $u(x, t_f)$ for $t_f = 0.02$. Discuss. (*Extra credit for comparing to the exact solution.*)

3. [Note that the stability of the explicit three-level scheme (2.98) for the heat equation is already addressed in section 2.12 of the textbook. The method is unconditionally unstable. It is the method I called the “leapfrog/Richardson” scheme in class; see the handout from the book by Körner.]

Show that the truncation error

$$T(x, t) = \frac{u(x, t + \Delta t) - u(x, t - \Delta t)}{2\Delta t} - \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{\Delta x^2}$$

of method (2.98) on page 35 satisfies

$$T(x, t) = \frac{1}{3}u_{ttt}(x, \tau)\Delta t^2 - \frac{1}{12}u_{xxxx}(\xi, t)\Delta x^2$$

for some $x - \Delta x \leq \xi \leq x + \Delta x$ and $t - \Delta t \leq \tau \leq t + \Delta t$. You will use the fact that $u_t(x, t) - u_{xx}(x, t) = 0$; recall $u(x, t)$ is the exact solution. (See assignment # 3 for the Taylor’s theorem analysis of the u_{xx} approximation.) Under the hypothesis that u_{tt} and u_{xxxx} are bounded, we see that $T(x, t) = O(\Delta t^2) + O(\Delta x^2)$. Thus this method is both more accurate than the simplest explicit method and completely useless.

4. Write a short MATLAB program to solve the PDE problem

$$u_t = u_{xx}, \quad u(0, t) = u(1, t) = 0, \quad u(x, 0) = x(1 - x)$$

using the simplest implicit method (2.63). Use MATLAB’s backslash command to solve the tridiagonal system and use sparse storage of the relevant matrix. Also, include the Fourier series for the exact solution to the problem into the program so that accuracy can be assessed. (See assignment # 3.)

Now use both $\Delta x = 0.1$ and $\Delta x = 0.05$ and explore the effect of taking large time steps Δt . In particular, produce a figure comparable to figure 2.4 which addresses the effect on accuracy of using large time steps in this unconditionally stable method.