Assignment #10 *revised*

Due *5:00pm Thursday May 12, 2005.*

Total of 52 points: 13 points each. Choose FOUR out of the given SIX.

1. ***revised.*** (13 *points*) In addition to its application to the potential equation, one can use Brownian motion to solve the heat equation

$$u_t = \frac{1}{2} u_{xx}. \tag{1}$$

(The constant $\frac{1}{2}$ is essential for equation (3) below.) Concretely, consider the problem: find $u(x,t)$ solving (1) for $-\infty < x < \infty$ and $t > 0$, subject to the initial condition

$$u(x,0) = f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and to the boundary conditions that $u(x,t) \to 0$ as $x \to \pm\infty$.

The exact solution to this problem, computed analytically, is

$$u(x,t) = \frac{1}{2} \operatorname{erfc}\left(\frac{x-1}{\sqrt{2t}}\right) - \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2t}}\right). \tag{2}$$

The definition of the *complementary error function* "erfc" is $\operatorname{erfc} x = 2\pi^{-1/2} \int_x^\infty e^{-t^2}\, dt$. The function erfc is built-in to Matlab so the solution can be calculated

```
>> u=0.5*erfc((x-1)/sqrt(2*t))-0.5*erfc(x/sqrt(2*t));
```

in Matlab, assuming `x`, `t` are defined. Plot this solution at several representative times; superimpose the plots and use `legend` to identify the plots.

The Brownian motion solution[1] of this problem

$$u(x,t) = \mathbb{E}\left(f(B_t)\right) \tag{3}$$

where B_t is the time t random position of a Brownian motion started at x. Here "$\mathbb{E}$" is the expectation. In other words, the solution $u(x,t)$ to (1) at the point (x,t) is found by averaging the values of the initial function f over the locations of the Brownian motion started at position x and run for time t.

Using the same ideas as in the programs `brown.m` and `brownpot.m` found on the course website, write a Matlab code which computes the solution by simulating a Brownian motion. In particular, consider $u(x, 0.1)$. Compare to the exact solution. Demonstrate convergence to the degree possible. (Convergence will be slow and noisy. I don't expect much. Don't spend a lot of computer time trying to get great accuracy.)

Extra Credit. ***revised.*** (3 *points*) Using the fact that $\rho(x,t) = (2\pi t)^{-1/2} e^{-x^2/2t}$ is the Green's function of equation (1), verify that (2) is indeed the solution of the given problem.

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[1] See a text on "stochastic calculus" for a proof of this Brownian formula. Note Brownian motion on the whole real line is easier to understand mathematically than on a finite interval.

2. (13 *points*)

a. Consider the linear two-point boundary value problem

$$u'' + f(x)u' + g(x)u = h(x), \qquad u(0) = a, \qquad u(1) = b, \tag{4}$$

where we are given continuous functions f, g, h and constants a, b. This is an *ODE* two-point boundary value problem. (For existence of a unique solution we need to add an assumption. It is standard to suppose $g(x) < 0$, for instance.)

One way to solve this problem is to do *linear shooting.* This method supposes that we can first solve the two initial value problems

$$u_1'' + f(x)u_1' + g(x)u_1 = h(x), \qquad u_1(0) = a, \qquad u_1'(0) = 0, \tag{5}$$

and

$$u_2'' + f(x)u_2' + g(x)u_2 = 0, \qquad u_2(0) = 0, \qquad u_2'(0) = 1. \tag{6}$$

We can solve these *initial value problems* numerically using `ode45`, for instance.

Show how we can then find a linear combination

$$u(x) = c_1u_1(x) + c_2u_2(x)$$

which satisfies the boundary value problem (4) *as long as* $u_2(1) \neq 0$. Explain with a picture why this method is called "shooting." (A reference for the linear shooting method is Burden & Faires, *Numerical Analysis.*)

b. Now solve the problem

$$u'' + (\sin x)u' - 5u = x^2 - 2, \qquad u(0) = 1, \qquad u(1) = -\pi \tag{7}$$

numerically by the linear shooting method, using `ode45`. Note that it will be useful, for plotting the solution, to specify a grid on which `ode45` reports its solution.

c. Now *explain* how you would use the above linear shooting method to do *semi-discretization in time* on the heat equation initial-boundary value problem

$$u_t = u_{xx} + F(x,t)u_x + G(x,t)u + H(x,t),$$
$$u(0,t) = A, \qquad u(1,t) = B, \qquad u(x,0) = u_0(x),$$

for F, G, H, u_0 continuous functions. "Semi-discretization in time" will involve a finite difference approximation for u_t but not for the spatial derivatives. Thus this is a form of the "method of lines" but with the lines in other direction, so to speak.

In particular, seek maximum accuracy by using a Crank-Nicolson type discretization of the time derivative; this is important for the problem to make sense. Rewrite the semi-discretized problem in the standard form of (4) above.

Use at least three complete sentences and write out a few equations in your explanation. No need to write code.

Extra Credit. (2 *points*) Problem (7) is not quite the same problem as solved in problem 3 on Assignment # 9, but it is close. Modify a code from that assignment to do (7) by a finite difference method and compare the result to what you got in part **b**.

Extra Credit. (3 *points*) Solve the boundary value problem (7) using MATLAB's built-in `bvp4c`. Compare to the result in part **b**.

3. (13 *points*) Solve the potential equation in a cube. In particular, solve

$$u_{xx} + u_{yy} + u_{zz} = 0$$

on the cube $(x, y, z) \in (0, 1) \times (0, 1) \times (0, 1)$ with boundary values

$$u(0, y, z) = \sin(\pi y) \sin(\pi z)$$

on the one face and $u = 0$ on remaining five faces.

Use the obvious centered finite difference approximation to set up equations for the unknowns $U_{j,k,l} \approx u(x_j, y_k, z_l)$. Draw a reasonable picture of the stencil of the method. Then write a program in MATLAB using $\Delta x = \Delta y = \Delta z = 1/J$ and also sparse linear algebra. Use `spy` to display the sparsity pattern of the matrix when $J = 5$; comment on this pattern.

You will not be able to do a lot of grid refinement, but use $J = 5, 10, 20, 30$ to show some convergence to the exact solution

$$u(x, y, z) = \frac{e^{\sqrt{2}\,\pi x} - e^{2\sqrt{2}\,\pi} e^{-\sqrt{2}\,\pi x}}{1 - e^{2\sqrt{2}\,\pi}} \sin(\pi y) \sin(\pi z).$$

4. Exercise 2.10 in MORTON & MAYERS. (13 *points*) [Note that exercise 2.3 was part of Assignment #3.]

5. Exercise 3.1 in MORTON & MAYERS. (13 *points*)

6. Exercise 4.8 in MORTON & MAYERS. (13 *points*)