Answers and Solutions to Final Exam

1. The formulas which result are on page 82 of FARLOW, with "L" instead of "c". Farlow calls them the "Euler formulas", which is (I think) a historically accurate name.

2. The answer is

$$u(x,t) = \frac{20}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\, e^{-3n^2\pi^2 t/25}\, \sin(n\pi x/5).$$

3. The answer is

$$u(x,y) = \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)}\, \frac{\sinh((2k+1)\pi(y-\pi))}{\sinh(-(2k+1)\pi^2)}\, \sin((2k+1)\pi x).$$

The following MATLAB program `membraneNo3.m` produces a 3D surface plot of $z = u(x,y)$:

```
% 3D plot of solution to problem 3 on final, Math 421 Fall 2007 (Bueler)

NN=20; % number of terms in sum
x = linspace(0,1,201);    y = linspace(0,pi,31);
[xx,yy] = meshgrid(x,y);

zz = zeros(size(xx));
c = zeros(1,NN);
for k=0:NN-1
  n = 2 * k + 1;
  c(k+1) = -4 / (pi * n);
  % direct calculation easily overflows:
  %   ff = sinh(n * pi * (yy-pi)) ./ sinh(n * pi*pi));
  small = exp(-2 * n * pi*pi);
  ff = ( exp(n*pi*(yy - 2 * pi)) - exp(-n*pi*yy) ) / (1 - small);
  zz = zz + c(k+1) * ff .* sin(n * pi * xx);
end

surf(x,y,zz,'LineStyle','none')
xlabel x, ylabel y, axis tight
```

4. **(a)** Separated solutions satisfy

$$\frac{\ddot{T}}{-T} = +\omega^2 = \frac{X''''}{X},$$

where the sign is chosen as in FARLOW with the intent of finding solutions which neither grow nor decay in time.

(b) Therefore

$$X(x) = A\cos\sqrt{\omega}x + B\sin\sqrt{\omega}x + C\cosh\sqrt{\omega}x + D\sinh\sqrt{\omega}x.$$

The boundary conditions give this system:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos\sqrt{\omega} & -\sin\sqrt{\omega} & \cosh\sqrt{\omega} & \sinh\sqrt{\omega} \\ \sin\sqrt{\omega} & -\cos\sqrt{\omega} & \sinh\sqrt{\omega} & \cosh\sqrt{\omega} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(Your may differ by multiplicative constants in each row.) The system fairly easily reduces to a smaller system for A and B (for instance), because the first two rows say $C = -A$ and $D = -B$, respectively. We get

$$\begin{bmatrix} -(\cos\sqrt{\omega} + \cosh\sqrt{\omega}) & -(\sin\sqrt{\omega} + \sinh\sqrt{\omega}) \\ (\sin\sqrt{\omega} - \sinh\sqrt{\omega}) & -(\cos\sqrt{\omega} + \cosh\sqrt{\omega}) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We seek *nontrivial* solutions A, B, C, D so that our eigenfunctions are nonzero. Thus the determinant hint, which gives

$$\left(\cos\sqrt{\omega} + \cosh\sqrt{\omega}\right)^2 + \sin^2\sqrt{\omega} - \sinh^2\sqrt{\omega} = 0.$$

Using identities $\cos^2 z + \sin^2 z = 1$ and $\cosh^2 z - \sinh^2 z = 1$ we get

$$\cos\sqrt{\omega} + \left(\cosh\sqrt{\omega}\right)^{-1} = 0.$$

(c) The answer is

$$\tilde{u}(x) = -\frac{\rho g}{24\alpha^2}\left(x^4 - 4x^3 + 6x^2\right).$$

(d) Here is a MATLAB program `beameig.m` which both plots the function we want to set to zero and finds its roots using `fzero`. (Note `fzero` is "Brent's method", if you want to understand it, but might as well be bisection.)

```
% script to find eigenvalues in problem 4 on M421 Final Exam Fall 07

g = @(z) cos(z) + 1 ./ cosh(z)

z=0:0.01:10;
plot(z,g(z)), grid on, xlabel('z=\omega^{1/2}')

format long g

zsoln = [fzero(g,1,2) fzero(g,4,5) fzero(g,7,8)]
%zsoln =
%          1.87510406871196   4.69409113297418   7.85475743823761

omsoln = zsoln.^2
%omsoln =
%          3.51601526850015   22.0344915646668   61.6972144135491
```

5. **(a)** Let $U(\xi,t) = \mathcal{F}_x[u](\xi,t)$ and $F(\xi) = \mathcal{F}_x[f](\xi)$. The problem transforms to

$$\begin{array}{ll} ODEs & U_{tt} = c^2(i\xi)^2 U = -c^2\xi^2 U, \\ ICs & \begin{array}{c} U(\xi,0) = F(\xi), \\ U_t(\xi,0) = 0. \end{array} \end{array}$$

The solutions of the ODEs are simple. They are all initial value problems of the form

$$\ddot{y} = -\alpha^2 y, \qquad y(0) = A, \qquad \dot{y}(0) = 0,$$

a problem which has solution $y(x) = A\cos(\alpha t)$. In fact,

$$U(\xi,t) = F(\xi)\cos(c\,\xi\,t).$$

Thus

$$u(x,t) = \mathcal{F}_\xi^{-1}[U](x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x}\,F(\xi)\cos(c\,\xi\,t)\,d\xi$$

or

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi y}\,f(y)\,dy\right)\cos(c\,\xi\,t)\,d\xi$$

(b) We must be able to Fourier transform $f(x)$. That is, $\int_{-\infty}^{\infty}|f(x)|\,dx < \infty$. In fact the same must apply to $u(x,t)$ and $U(\xi,t)$; f, u, and U must all be absolutely integrable.

(c) Continuing from the result in **(a)** we change order of integration to get

$$u(x,t) = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi(y-x)}\,\cos(c\,\xi\,t)\,d\xi\right] f(y)\,dy.$$

Thus

$$W(t,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi\,z}\,\cos(c\,\xi\,t)\,d\xi.$$

Unfortunately there is no reason to believe that this integral converges because the integrand shows no decay at all.

(d) $u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$.

(e) The formula in part **(c)** turns out to be meaningful as a pair of delta functions. The fundamental fact is that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi\,z}\,d\xi = \delta_0(z).$$

Thus, using $\cos\theta = \frac{1}{2}\left[e^{i\theta} + e^{-i\theta}\right]$, we have

$$W(t,z) = \frac{1}{2}\left[\delta_0(z+ct) + \delta_0(z-ct)\right],$$

and so the integral formula for $u(x,t)$ in part **(c)** reduces to the D'Alembert result in part **(d)**.

I. **(a)** If $u(x,t) = T(t)X(x)$ then

$$\ddot{T}X + \gamma\dot{T}X = c^2TX''.$$

Choosing the sign of the constant the same way as in the wave and heat equation, we have

$$\frac{\ddot{T} + \gamma\dot{T}}{c^2T} = -\lambda^2 = \frac{X''}{X}.$$

Incorporating the boundary conditions we see that the eigen-problem for $X(x)$ is familiar,

$$X'' + \gamma^2 X = 0, \qquad X'(0) = 0, \qquad X'(\pi) = 0,$$

so

$$X_n(x) = \cos nx, \qquad \lambda_n = n, \qquad n = 0, 1, 2, \ldots$$

As noted, the ODE for $T(t)$ is new:

$$\ddot{T} + \gamma\dot{T} + c^2n^2T = 0.$$

This can be solved by substituting $T(t) = e^{rt}$ and finding the characteristic equation

$$r^2 + \gamma r + c^2n^2 = 0$$

and its roots

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4c^2n^2}}{2}.$$

(When $n = 0$ we get $T_0(t) = a_0/2 + b_0e^{-\gamma t}$.)

Once case is when $\gamma^2 - 4c^2n^2 = 0$, that is $\gamma = 2cn$, but we are instructed to ignor it (and it is rare, in some senses anyway).

But there are the two major cases where $\gamma > 2cn$ and $\gamma < 2cn$; the former only occurs for finitely many n while the latter always occurs for infinitely many n:

$$T_n(t) = e^{-(\gamma/2)t}\begin{cases} a_n\cosh((1/2)\sqrt{\gamma^2 - 4c^2n^2}\ t) + b_n\sinh((1/2)\sqrt{\gamma^2 - 4c^2n^2}\ t), & \gamma > 2cn, \\ a_n\cos((1/2)\sqrt{4c^2n^2 - \gamma^2}\ t) + b_n\sin((1/2)\sqrt{4c^2n^2 - \gamma^2}\ t), & \gamma < 2cn, \end{cases}$$

except that

Thus the solution of the PDE and BCs is

$$u(x,t) = T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos nx.$$

If we incorporate the initial conditions and split the sum at the point $\gamma > 2cn$ then we have

$$\begin{aligned} u(x,t) = \frac{a_0}{2} &+ \sum_{n<\gamma/(2c)} a_ne^{-(\gamma/2)t}\cosh((1/2)\sqrt{4c^2n^2 - \gamma^2}\ t)\cos nx \\ &+ \sum_{n>\gamma/(2c)} a_ne^{-(\gamma/2)t}\cos((1/2)\sqrt{4c^2n^2 - \gamma^2}\ t)\cos nx \end{aligned}$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

(b) If $\gamma = 0$ then the case "$n < \gamma/(2c)$" does not occur. The sum reduces to

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(cnt) \cos nx$$

and the formula for a_n is the same. We can check that $u_{tt} = c^2 u_{xx}$ and that the same boundary and initial conditions apply.

(c) If $\gamma > 0$ then every term in the sum except the constant term decays to zero. Thus the steady state is

$$\tilde{u}(x) = \frac{a_0}{2} = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx.$$

(d) The limiting case is where γ is large. Concretely, $\gamma \to \infty$ and $c \to \infty$ with c^2/γ going to some positive limit K, in which case we get the heat equation $u_t = K u_{xx}$. But it is hard to see the solution to this heat equation in the solution we wrote down for (a).

II. (a) Note that for an infinite slab of thickness $2L$, if everything is symmetric there is only heat flow only through the thickness of the slab and not in the lateral directions. Thus we may think of any rod across its thickness as insulated from its adjacent, parallel rods.

(b) The answer is

$$u(x,t) = \frac{4\theta_0}{\pi} \sum_{n>0\,\mathrm{odd}} \frac{1}{n} e^{-n^2\pi^2 Kt/(2L)^2} \sin(n\pi x/(2L)).$$

(c) First,

$$u_x(x,t) = \frac{2\theta_0}{L} \sum_{n>0\,\mathrm{odd}} e^{-n^2\pi^2 Kt/(2L)^2} \cos(n\pi x/(2L))$$

so the equation to solve is

$$\Phi_0 = F(Kt) \qquad \text{where} \qquad F(z) = \frac{2\theta_0}{L} \sum_{m=0}^{\infty} e^{-(2m+1)^2\pi^2 z/(2L)^2}.$$

For $z = Kt > 0$, this sum can be rigorously shown to converge by application of the ratio test, but one should expect that it converges because it is a sum of rapidly decaying exponentials (as functions of m).

(d) We can truncate the sum and then numerically solve the equation, as in the MATLAB program `ageEarth.m` below. The result I get for the "physically reasonable" material values pulled off various websites is about 53 million years. *And Darwin recognized this as too* small *an age.*

```
% solve  Phi0 = F(Kt)  to find the age of the earth, where
%     F(z) = (2 theta0/L) sum_{m=0}^infty e^{-(2m+1)^2 pi^2 z/(2L)^2}

MM = 501;  % highest integer in sum; can show a few hundred terms needed
           % to get reasonably close to infinite sum

%%% reasonable constants
L = 6357.0e3;  % m;      source: "earth radius" wikipedia page
K = 1.0e-6;    % m^2/s; "typical value for rock"
               %         source: "thermal diffusivity" wikipedia page
theta0 = 1811; % K; melting point of iron
               %         source: "melting points of the elements" wikipedia page
Phi0 = 0.025;  % K/m; "geothermal gradient in crust"    source:
               %    http://gpc.edu/~pgore/Earth&Space/GPS/earthinterior.html
Kt = 10.^(7:0.1:12);

% %%% fake constants
% L = 1;  theta0 = 1;  K = 1;  Phi0 = 1;
% Kt = 0:0.01:1;
% %%% solution for t about 0.282 when MM = 5 and 0.281 when MM = 1

ss = zeros(size(Kt));
for m=1:2:MM
   ss = ss + exp(- m * m * pi * pi * Kt / (4 * L * L));
end
FF = (2 * theta0 / L) * ss;

loglog(Kt, Phi0*ones(size(Kt)), '--', Kt, FF)
axis([min(Kt) max(Kt) 1e-3 1])
xlabel('Kt'), ylabel('temperature gradient')
legend('\Phi_0', 'F(Kt)')

Kt_cross = 10^9.22;  % result from blowing up the log-log graph
t_cross = Kt_cross / K;
t_cross_year = t_cross / 31556926  % divide by number of seconds in a year
```