# Selected Solutions to Assignment #9

**Exercise 2.** **(a)** & **(c)**

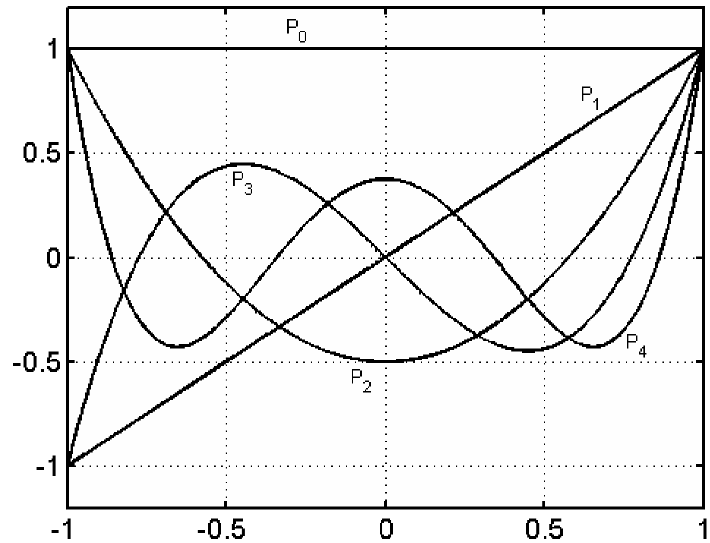


FIGURE 1. The first five Legendre polynomials $P_0(x), \dots, P_4(x)$.

**(b)** The four facts stated in the problem about $P_4$ become these facts when one expands the inner products in the problem statement:

$$0 = \int_{-1}^{1} 1\, P_4(x)\, dx = \frac{2}{5} + 0\, c_3 + \frac{2}{3}\, c_2 + 0\, c_1 + 2\, c_0$$

$$0 = \int_{-1}^{1} x\, P_4(x)\, dx = 0 + \frac{2}{5}\, c_3 + 0\, c_2 + \frac{2}{3}\, c_1 + 0\, c_0$$

$$0 = \int_{-1}^{1} \frac{1}{2}(3x^2 - 1)\, P_4(x)\, dx = \frac{8}{35} + 0\, c_3 + \frac{4}{15}\, c_2 + 0\, c_1 + 0\, c_0$$

$$0 = \int_{-1}^{1} \frac{1}{2}(5x^3 - 3x)\, P_4(x)\, dx = 0 + \frac{4}{35}\, c_3 + 0\, c_2 + 0\, c_1 + 0\, c_0.$$

The coefficients above are found by integration. For example, because $P_1(x) = x$, the second line above involves five integrals

$$\int_{-1}^{1} x\, 1\, dx = 0, \quad \int_{-1}^{1} x\, x\, dx = \frac{2}{3}, \quad \int_{-1}^{1} x\, x^2\, dx = 0, \quad \int_{-1}^{1} x\, x^3\, dx = \frac{2}{5}, \quad \text{and} \quad \int_{-1}^{1} x\, x^4\, dx = 0.$$

I set up the four facts as a linear system $A\mathbf{x} = \mathbf{b}$ and solve it in Octave (like MATLAB, but free) and get $c_0 = 0.085714285714286$, $c_1 = 0$, $c_2 = -0.857142857142857$, and $c_3 = 0$.

A little experimentation shows that $c_0 = 3/35$ and $c_2 = -6/7$. Thus $f(x) = x^4 - (6/7)x^2 + (3/35)$ is a multiple of $P_4(x)$. Dividing by $f(1) = 8/35$ gives

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right).$$

**Exercise 3.** **(a)** Note for (near) future use that $\overline{e^{iy}} = e^{-iy}$ and that $e^{-iy} = \cos y - i \sin y$.

To find the coefficients we multiply on by $(2\pi)^{-1/2}\, e^{imx}$ and integrate, but we do it on the *left*, and we interpret it all as inner products and use orthonormality:

$$\left(\frac{1}{\sqrt{2\pi}}e^{imx}, f\right) = \left(\frac{1}{\sqrt{2\pi}}e^{imx}, \sum_{n=-\infty}^{\infty} a_n \frac{1}{\sqrt{2\pi}}e^{inx}\right) = \sum_{n=-\infty}^{\infty} a_n \left(\frac{1}{\sqrt{2\pi}}e^{imx}, \frac{1}{\sqrt{2\pi}}e^{inx}\right)$$
$$= \sum_{n=-\infty}^{\infty} a_n\, \delta_{nm} = a_m.$$

For $f(x) = |x|$ we can write out the integral; at this point I use the conjugate of $e^{iy}$:

$$a_m = \left(\frac{1}{\sqrt{2\pi}}e^{imx}, f\right) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-imx}\, |x|\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (\cos mx - i \sin mx)\; |x|\, dx.$$

I have written out the complex exponential only because after a little comparison I think this leads to the easiest computation. In fact, the integral of $|x| \sin mx$ is zero because $\sin mx$ is odd and $|x|$ is even. Using the evenness of $|x| \cos mx$, we have an integral to do by parts:

$$a_m = \frac{2}{\sqrt{2\pi}} \int_0^{\pi} x \cos mx\, dx = \sqrt{\frac{2}{\pi}}\frac{1}{m}\left(x \sin mx\Big]_0^{\pi} - \int_0^{\pi} \sin mx\, dx\right)$$
$$= -\frac{1}{m^2}\sqrt{\frac{2}{\pi}}\Big[-\cos mx\Big]_0^{\pi} = \frac{1}{m^2}\sqrt{\frac{2}{\pi}} \begin{cases} 0, & m \text{ even}, \\ -2, & m \text{ odd}. \end{cases}$$

This calculation only holds for $m \neq 0$. For $m = 0$ we have

$$a_0 = \frac{2}{\sqrt{2\pi}} \int_0^{\pi} x\, dx = \frac{2}{\sqrt{2\pi}} \frac{\pi^2}{2} = \frac{\pi^2}{\sqrt{2\pi}}.$$

**(b)** Collecting all this together we have the complex Fourier series

$$|x| = \sum_{n=-\infty}^{\infty} a_n \frac{1}{\sqrt{2\pi}}e^{inx} = \frac{\pi^2}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} + \sum_{n \text{ odd}} \left(\frac{1}{n^2}\sqrt{\frac{2}{\pi}}(-2)\right) \frac{1}{\sqrt{2\pi}}e^{inx} = \frac{\pi}{2} - \sum_{n \text{ odd}} \frac{2}{\pi n^2} e^{inx}.$$

The sum is over all odd integers $\cdots < -5 < -3 < -1 < 1 < 3 < \ldots$.

We want to write this as a *cosine* series. The point is that the coefficient of $e^{inx}$ is the same as the coefficient of $e^{i(-n)x}$ in this case. Thus we can collect pairs $n$, $-n$, identify the cosine, and rewrite the sum:

$$
\begin{aligned}
|x| &= \frac{\pi}{2} - \sum_{n=1,3,5,\ldots} \frac{2}{\pi n^2}\left(e^{inx} + e^{-inx}\right) = \frac{\pi}{2} - \sum_{n=1,3,5,\ldots} \frac{4}{\pi n^2}\cos nx \\
&= \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)^2}\cos(2k+1)x.
\end{aligned}
$$

We have a cosine series. (*There is no claim that this was the most efficient route to the cosine series, but it does show that the complex exponential series is a fairly universal route to sine and cosine series.*

**(c)** The plot (*not shown*) of $\bar{f}(x)$ is a sawtooth wave and the plot of $\bar{f}'(x)$ is a square wave. The latter has magnitude 1 and period $2\pi$ and is an odd function.

**(d)** Differentiating the result of part **(b)** gives the Fourier sine series:

$$
\bar{f}'(x) = \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)}\sin(2k+1)x.
$$

By contrast, the result in exercise #3 of Lesson 5 of Farlow is the sine series

$$
1 = \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)}\sin(2k+1)\pi x.
$$

But of course on the interval $(0,\pi)$, $\bar{f}'(x) = 1$.

We see that the only difference between the two series is that in the first, for $x$ in the interval $(0,\pi)$ we get 1 from the sine series, while in the second statement $x$ is on the interval $(0,1)$. Of course the replacement of "$x$" in the first series by "$\pi x$" in the second shows they are the same series.

In either case these series are odd. Therefore they cannot be equal to one *everywhere*, and they are not. They are sine series for a square wave oscillating between 1 and $-1$.

Partial sums of these sine series are an excellent illustration of the Gibbs effect.

**Exercise 4.** **(a)** *Easy, and omitted.*

**(b)** Do the integrals

$$
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x/2)|\,\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin(x/2)\,\cos nx\,dx
$$

perhaps using the trig identity

$$
\sin(x/2)\cos nx = \frac{1}{2}\left[\sin((n+(1/2))x) - \sin((n-(1/2)x)\right].
$$

Get

$$
a_n = -\frac{4}{\pi(4n^2-1)}
$$

for $n = 0, 1, 2, 3, \ldots$. Therefore

$$|\sin(x/2)| = \frac{2}{\pi} - \sum_{n=1}^{\infty} \frac{4}{\pi(4n^2 - 1)} \cos nx.$$

**(c)** & **(d)** The periodic extension is continuous. The periodic extension is smooth except at even multiples of $\pi$. So the convergence should be poorest at even multiples of $\pi$, and this is exactly what we see in the graphs of the fifth and ninth partial sums.