Selected Solutions to Assignment #8

Lesson 6, #1 (*The answer in the back of the book is clearly wrong. Also, this problem is one you cannot complete unless you know α and h numerically. On the other hand it is a good, realistic, modeling problem for heat in a rod! So here goes:*)

The problem as it stands has non-homogeneous boundary values. We transform them to homogeneous ones by defining

$$S(x) := \left(\frac{1-h}{1+h}\right) x + 1,$$

the steady state solution, and by removing that from the time-dependent temperature:

$$w(x,t) := u(x,t) - S(x) = u(x,t) - \left(\frac{1-h}{1+h}\right) x - 1.$$

Now we can transform the entire problem

$$\begin{array}{lc} PDE & w_t = \alpha^2 w_{xx} \\ BCs & \begin{array}{c} w(0,t) = 0, \\ w_x(1,t) + hw(1,t) = 0 \end{array} \\ IC & w(x,0) = \sin(\pi x) + x - \left(\frac{1-h}{1+h}\right) x - 1 \end{array}$$

The PDE BVP part of this problem for $w(x,t)$ is exactly the one which is addressed in Lesson 7. In particular, the eigenvalues λ_n are the positive solutions to

$$\tan\lambda = -\frac{\lambda}{h} \tag{1}$$

and the eigenfunctions are

$$X_n(x) = \sin(\lambda_n x).$$

The general solution to the PDE BVP for $w(x,t)$ is

$$w(x,t) = \sum_{n=1}^{\infty} B_n\, e^{-\lambda_n^2 \alpha^2 t}\, \sin(\lambda_n x). \tag{2}$$

To satisfy the IC we need to have an orthogonality condition

$$\int_0^1 \sin(\lambda_m x)\, \sin(\lambda_n x)\, dx = 0, \qquad m \neq n.$$

The textbook should emphasize in Lesson 7 that we must have this fact true. It follows from the Sturm-Liouville theorem. (I will not prove it, but I will state it clearly in class.)

By multiplying each side of (2) by $\sin(\lambda_m x)$ and integrating from 0 to 1 we get this formula for the constants B_n:

$$(3) \qquad B_n = \left(\int_0^1 \sin^2(\lambda_n x)\, dx \right)^{-1} \int_0^1 \left[\sin(\pi x) + x - \left(\frac{1-h}{1+h} \right) x - 1 \right] \sin(\lambda_n x)\, dx$$

We should stop here. In practice, the solution of the original problem is found this way:

- (i) Specify particular values for α and h.
- (ii) Solve (1) numerically, for instance by bisection, for the first N solutions.
- (iii) Compute the integrals (3) numerically for $n = 1, 2, \ldots, N$. Use the numerical values from step (i) as needed.
- (iv) All numerical constants in equation (2) are now known concretely, though only approximately. Of course the infinite sum now stops at N.
- (v) Write the formula for $u(x,t)$ by undoing the substitutions which defined $w(x,t)$ and $S(x)$.
- (vi) Display $u(x,t)$ as a movie. (*That is, actually see* a solution *to the original problem, not just a bunch of formulas.*)

If you are curious, I can show how the final numerical errors in the computation of $u(x,t)$, *for practical computers and a short amount of computer time*, can be made less than 10^{-10}, for example.

Lesson 7, #4 The solution is a list of eigenvalues λ_n and eigenfunctions $X_n(x)$ for $n = 0, 1, 2, \ldots$:

$$\lambda_n = -\pi^2 n^2 \qquad \text{and} \qquad X_n(x) = \cos(n\pi x).$$

This includes the $n = 0$ *case!* That is,

$$\lambda_0 = 0 \qquad \text{and} \qquad X_0(x) = 1.$$

The lowest frequency eigenfunction has zero frequency but it is non-zero. It is a constant.

Lesson 7, #3 (*The solution in the back of the book is (slightly) wrong. The sum should only be over* odd n.)

Now that we have found the eigenstuff, we can go right to the expansion of the solution and the satisfaction of boundary conditions. The time-dependent general solution PDE BVP is

$$u(x,t) = B_0 + \sum_{n=1}^{\infty} B_n\, e^{-n^2\pi^2 t} \cos(n\pi x).$$

The constants $\{B_n\}$ must be chosen to satisfy the initial condition:

$$x = u(x,0) = B_0 + \sum_{n=1}^{\infty} B_n \cos(n\pi x).$$

This Fourier cosine series can be looked up (!) or you can do these integrals, which aren't very hard:

$$\int_0^1 x\cos(n\pi x)\,dx = \begin{cases} 0, & n>0 \text{ even}, \\ -\frac{2}{n^2\pi^2}, & n \text{ odd}, \\ \frac{1}{2}, & n=0, \end{cases}$$

$$\int_0^1 \cos^2(n\pi x)\,dx = \begin{cases} \frac{1}{2}, & n>0, \\ 1, & n=0. \end{cases}$$

You will get

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^2}\, e^{-n^2\pi^2 t}\, \cos(n\pi x).$$

Lesson 8, #1 (*The solution in the back of the book is again wrong! There should be an exponential decay in time factor inside the sum and there is an additional sign error; see the correct solution below.*)

This is a conduction-advection problem of the type discussed in the lesson, with $v = 1$. Use the transformation

$$u(x,t) \overset{\circledast}{=} e^{(x-t/2)/2}\, w(x,t).$$

This defines a new function $w(x,t)$. We have a new PDE IBVP:

$$\begin{array}{ll} PDE & w_t = w_{xx}, \\ BCs & \begin{array}{l} w(0,t) = 0, \\ w(1,t) = 0, \end{array} \\ IC & w(x,0) = 1. \end{array}$$

(*There is a bit of error-prone calculation to get here. Think it all through!*)

This is a PDE IBVP we have seen before, namely exercise #4 in Lesson 5. We can and should re-use such solutions. The solution in the back of the book actually *is* correct for Lesson 5 #4:

$$w(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\, e^{-\pi^2(2n-1)^2 t}\, \sin((2n-1)\pi x).$$

Going back to $u(x,t)$ by the transformation $\circledast$ gives the solution

$$u(x,t) = \frac{4}{\pi} e^{(x-t/2)/2} \sum_{n=1}^{\infty} \frac{1}{2n-1}\, e^{-\pi^2(2n-1)^2 t}\, \sin((2n-1)\pi x).$$