Selected Solutions to Assignment #7

Lesson 5, #2 First note that the given trigonometric identity can be derived from these two famous facts:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

In particular, subtracting the first equation from the second gives

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B).$$

Now insert $A = mx$ and $B = nx$ to get the stated identity.

Now we integrate:

$$\int_0^1 \sin(m\pi x)\, \sin(n\pi x)\, dx = \frac{1}{2}\int_0^1 \cos((m-n)\pi x) - \cos((m+n)\pi x)\, dx$$
$$\overset{\star}{=} \frac{1}{2\pi}\left[\frac{\sin((m-n)\pi x)}{m-n} - \frac{\sin((m+n)\pi x)}{m+n}\right]_0^1 = 0$$

because $\sin(0) = 0$ and $\sin(k\pi) = 0$ if k is an integer.

The exception is if $m = n$. In this case step $\star$ is invalid because we would divide by zero. In fact, we have

$$\int_0^1 \sin^2(m\pi x)\; dx = \frac{1}{2}\int_0^1 \cos(0) - \cos(2m\pi x)\, dx = \frac{1}{2}$$

in this case, as also indicated in class.

Lesson 5, #5 The point of this problem is that we can immediately write the solution of the whole problem—that is, of the whole IBVP—as a sum of separated solutions:

$$u(x,t) = e^{-2^2\pi^2 t}\sin(2\pi x) + \frac{1}{3}e^{-4^2\pi^2 t}\sin(4\pi x) + \frac{1}{5}e^{-6^2\pi^2 t}\sin(6\pi x).$$

This is because we already have the initial condition as a linear combination of the eigenfunctions $\sin(n\pi x)$. In any case, one can check that this formula solves all parts of problem 4.

Lesson 6, #2 The steady state of this problem is $\bar{u} = x$. Therefore define

$$w(x,t) := u(x,t) - x.$$

Then w solves this IBVP:

$$\begin{gathered} w_t = w_{xx} \\ w(0,t) = w(1,t) = 0 \\ w(x,0) = x^2 - x \end{gathered}$$

Now the separated solutions are completely familiar: $w(x,t) = T(t)X(x)$ where $X(x) = \sin(n\pi x)$ and $T(t) = e^{-n^2\pi^2 t}$. Thus

$$w(x,t) = \sum_{n=1}^{\infty} B_n\, e^{-n^2\pi^2 t}\, \sin(n\pi x).$$

Furthermore, integrating twice by parts,

$$\begin{aligned} B_n &= 2\int_0^1 (x^2 - x)\sin(n\pi x)\, dx = \frac{2}{n\pi}\int_0^1 (2x-1)\cos(n\pi x)\, dx \\ &= -\frac{2}{n^2\pi^2}\int_0^1 2\sin(n\pi x)\, dx = +\frac{4}{n^3\pi^3}\left[\cos(n\pi) - 1\right] \\ &= \frac{4}{n^3\pi^3}\begin{cases} -2, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases} \end{aligned}$$

It follows that

$$u(x,t) = x - \sum_{k=0}^{\infty} \frac{8}{(2k+1)^3\pi^3} e^{-(2k+1)^2\pi^2 t}\, \sin((2k+1)\pi x),$$

which actually matches the formula in the back of the textbook.