

Selected Solutions to Assignment #6

Lesson 1, #2 *The point of this exercise is that some solutions of the standard heat equation can be found by simple substitution. Though we find the solutions we find from separation of variables, it is not clear here how to make the solutions satisfy boundary conditions.*

Simply substitute $u(x, t) = e^{ax+bt}$ into $u_t = \alpha^2 u_{xx}$:

$$be^{ax+bt} = \alpha^2 a^2 e^{ax+bt}.$$

Because e^z is never zero, a and b must satisfy

$$b = \alpha^2 a^2.$$

Note a, b can be complex. Thus this is a family of solutions:

$$u(x, t) = e^{\alpha^2 a^2 t} e^{ax}.$$

That is, the heat equation has an exponential rate in time which is the square of the frequency in space.

In fact we find that, because of boundary conditions, a is purely imaginary so a^2 is negative.

Lesson 1, #5 We have an unknown function $u(x, y)$ for which the mixed derivative is zero. So we know the y derivative of u_x is zero. Thus

$$u_x(x, y) = f(x).$$

But then u is a function with known x derivative, namely $f(x)$. But this means that u might depend on y additively:

$$u(x, y) = \int f(x) dx + g(y) = F(x) + g(y).$$

Here $F(x)$ is a function with derivative $f(x)$.

You can check that if $u(x, y) = F(x) + g(y)$ for any pair of differentiable functions F and g then $u_{xy} = 0$. Thus F and g are completely arbitrary.

Lesson 2, #2 The steady state temperature distribution is the one for which $u_t = 0$. This steady state distribution is a function of x only. Let $w(x)$ be the steady state. It satisfies:

$$0 = \alpha^2 w'' + 1, \quad w(0) = 0, \quad w(1) = 1.$$

One can find the general solution of this ODE by integration. It is a trivial ODE because we are *told* the derivative of a function, not just that there is a relation between the function and its derivatives. In fact,

$$w''(x) = -\alpha^{-2}$$

so

$$w'(x) = -\alpha^{-2}x + c_0$$

and

$$w(x) = -0.5\alpha^{-2}x^2 + c_0x + c_1.$$

The boundary conditions give the two equations

$$\begin{aligned} 0 &= c_1, \\ 1 &= -0.5\alpha^{-2} + c_0 + c_1. \end{aligned}$$

It follows that $c_0 = 1 + 0.5\alpha^{-2}$ and that

$$w(x) = -0.5\alpha^{-2}x^2 + (1 + 0.5\alpha^{-2})x.$$

This temperature distribution is *the* concave down parabola with second derivative $-\alpha^{-2}$ which goes through $w(0) = 0$ and $w(1) = 1$. In particular, if α^2 is very large, for instance if the conductivity of the rod is high, then this parabola is very nearly the straight line between $w(0) = 0$ and $w(1) = 1$, that is $w(x) = x$.

Lesson 3, #1 The boundary conditions for the problem stated in (3.6) are that the left end of the rod is insulated while the right end is gaining or losing heat so that the temperature at the right end converges toward 20°C . Your sketches should satisfy these conditions.

The steady state distribution is a function $w(x)$ satisfying $0 = \alpha^2 w''$, $w'(0) = 0$, and $w'(200) = -(h/k)[w(200) - 20]$. The differential equation says w is a straight line:

$$w(x) = ax + b.$$

But the left boundary condition says $a = 0$, which means w is constant. But the right boundary condition, $0 = -(h/k)[w(200) - 20]$, says $w(200) = 20$, so

$$w(x) = 20.$$

The steady state distribution is that the whole rod is at 20°C .

This result *is* fairly obvious to my intuition. We have an object in contact with a 20°C pool of water, but the object is otherwise insulated and there are no heat sources within the object. Regardless of its initial temperature distribution, it is clear that all parts of this object will end up at 20°C .