Selected Solutions to Assignment #10

Exercise 1. The function is piecewise continuous and its 2π periodic extension is continuous. Also, left- and right-sided derivatives exist at *every* point. Thus the theorem on page 90, which is Dirichlet's theorem, applies at every point. It shows that the Fourier series converges at every point in $[-\pi, \pi]$.

Exercise 2. (*My main goal here is to explain the word "*correspondence*" in this context.*)

We have a Fourier cosine series for the function $f(x) = x$ on the interval from 0 to π. But cosines are even, so this must be the Fourier cosine series for $|x|$ on the interval from $-\pi$ to π. And the cosines are 2π periodic. Thus the question is whether this Fourier cosine series converges to the even 2π-periodic extension of $f(x) = x$ on the interval 0 to π.

There is a stage where we find the coefficients a_n in the Fourier cosine series. All we know at that stage is that the coefficients came from certain integrals of $|x|$ (or x, depending on how you did the integral). This is the "correspondence". We don't yet know that there is *equality* between the function $f(x) = x$ and its Fourier series. Equality is a question of whether the Fourier series converges, and if so, what the Fourier series converges to. We only learned about equality by proving Dirichlet's theorem.

In this case, however, the even 2π-periodic extension of $|x|$ is continuous and piecewise smooth. (It is an easily visualized kind of sawtooth wave.) The corollary on page 92 shows that the Fourier series converges to $|x|$ for all x in $[-\pi, \pi]$. In particular, the Fourier series converges to x for all x in $[0, \pi]$.

Now we have equality:

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

for all x in $[0, \pi]$. We can exploit this equality any way we want. In particular, we can evaluate each side at $x = 0$. We get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

which turns into the desired (and *not* obvious) sum.

Exercise 3. *Same ideas as in* **2**, *but applied to more famous series.*

Exercise 4. Consider $x^{2/3}$ restricted to $[-\pi, \pi]$ and then periodically extended to the whole real line. This function is graphed in Figure 1. It is continuous.

The behavior at even multiples of π is very different from that at odd multiples. In fact,

$$\frac{d}{dx}(x^{2/3}) = \frac{2}{3}\,\frac{1}{x^{1/3}}.$$

Therefore the derivative is not defined at $x = 0$. In addition, the derivative is unbounded near $x = 0$. Thus the derivative is not piecewise continuous. The theorem on page 90, which is

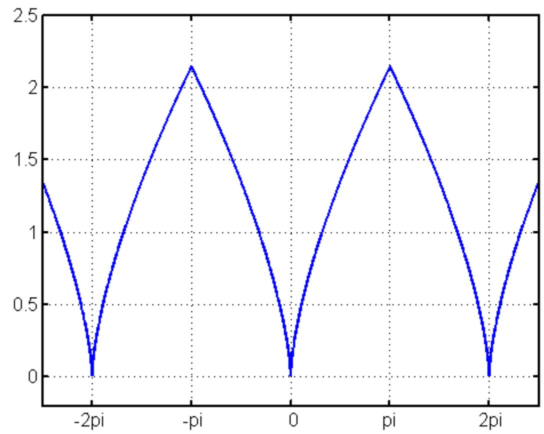

FIGURE 1. The 2π-periodic extension of $x^{2/3}$ defined on $[-\pi, \pi]$.

Dirichlet's theorem, does not tell us that the Fourier series converges at $x = 0$ because that is a place where the one sided derivatives $f'_L(0)$ and $f'_R(0)$ do not exist.

At the odd multiples of π, the one sided derivatives do exist. In fact, Dirichlet's theorem applies at every point except at even multiples of π, and shows that the Fourier series converges to the correct value (i.e. to the value of the 2π-periodic extension of $x^{2/3}$).

In fact I think the Fourier series converges at even multiples of π but I do not know and I do not know how to prove it one way or the other. In any case, I am confident that the Fourier series converges very slowly there.

Exercise 6. **(a)** The eigenfunctions are $X_n(x) = \cos nx$, $n = 0, 1, 2, \ldots$, satisfying $X'' + \lambda^2 X = 0$, $X'(0) = 0$, and $X'(\pi) = 0$, with eigenvalues $\lambda_n = n$. The separated solutions are $u(x,t) = T_n(t)X_n(x)$ where $T_n(t) = e^{-\lambda_n^2 t} = e^{-n^2 t}$. Therefore the general solution to the PDE and BCs is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx.$$

We find the coefficients a_n by noting that

$$u(x,0) = \phi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

Either using orthogonality ourselves, or noting that this is the problem of finding the classical Fourier cosine series for the even periodic extension of $\phi(x)$, we have this formula for the coefficients:

$$a_n = \frac{2}{\pi} \int_0^{\pi} \phi(x) \cos nx \, dx.$$

(b) Now that we have a particular function $\phi(x)$ we can find each a_n and then sketch $u(x,t)$ for some different values of t. First, for $n \geq 1$,

$$a_n = \frac{2}{\pi}\left(-\int_0^{\pi/2} \cos nx \, dx + \int_{\pi/2}^{\pi} \cos nx \, dx \right) = \begin{cases} 0, & n \text{ even}, \\ \frac{4(-1)^k}{\pi(2k-1)}, & n = 2k-1 \text{ odd}. \end{cases}$$

Separately we see $a_0 = 0$; note this is because the rod is insulated so the heat energy is conserved, but on the other hand the average initial temperature is zero. Thus

$$u(x,t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} e^{-(2k-1)^2 t} \cos(2k-1)x. \tag{1}$$

For our sketch, at $t = 0$ we recover $\phi(x)$. On the other hand, the steady state is $u(x, t = +\infty) = 0$ because, looking at our solution $u(x,t)$, *all modes decay to zero as* $t \to \infty$. In fact the sketch can be recovered by hand, but it is easier for me to put here by using MATLAB.

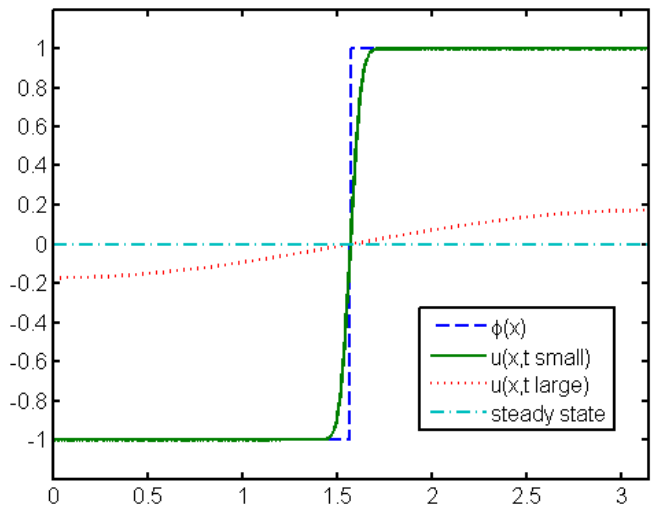

FIGURE 2. Graphs of $u(x,t)$ in exercise **3 (b)** for $t = 0$, small t [$t = 0.001$], large t [$t = 2$], and the steady state.

(c) The even periodic extension $\tilde{\phi}(x)$ of $\phi(x)$ is continuous on the real line except for jumps at odd multiples of $\pi/2$. It is constant on each interval between jumps, so it is piecewise smooth. It happens that the even periodic extension satisfies

$$\tilde{\phi}(x) = \frac{\tilde{\phi}(x+) + \tilde{\phi}(x-)}{2}$$

at *every* point because I defined $\phi(\pi/2)$ to be "mid-way" in the jump. Thus Dirichlet's theorem, namely Corollary 9.2 in the handout, shows that the Fourier series of $\phi(x)$ converges at *every* x.

On the other hand, formula (1) reduces to the Fourier series for $\phi(x)$ when $t = 0$. Thus (1) converges to $\phi(x)$ when $t = 0$.

(d) The partial sums of the Fourier series for $\phi(x)$ show the Gibbs effect at the jumps. This is because the coefficients a_n in the Fourier series for $\phi(x)$ decay slowly as a function of n (and they *must* decay slowly because the jump is a high frequency effect, but this run-on sentence is becoming a circular argument ...). When we use a small positive t in formula (1), however, then for sufficiently large n the coefficients a_n decay quite fast because of the $e^{-n^2 t}$ multiplier.

So the Gibbs effect goes away and we get a smooth function that approximates $\phi(x)$ arbitrarily well when we make t sufficiently small.

Exercise 7. **(a)** *Easy*, and already used in **6(a)**.

(b) (*Here is the go-back-to-the-definition-and-calculate method. I also accepted a graphical argument.*)

Suppose $f(x)$ is even. By definition, using the evenness of $f(x)$ in step $\circledast$,

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} \overset{\circledast}{=} \lim_{h\to 0} \frac{f(x-h) - f(x)}{h} = -\lim_{h\to 0} \frac{f(x-h) - f(x)}{-h}.$$

But the limit is two-sided. That is,

$$\lim_{h\to 0} G(h) = \lim_{-h\to 0} G(h).$$

So we can finish up by defining a new variable $\hat{h} := -h$, getting

$$f'(-x) = -\lim_{h\to 0} \frac{f(x-h) - f(x)}{-h} = -\lim_{\hat{h}\to 0} \frac{f(x+\hat{h}) - f(x)}{\hat{h}} = -f'(x).$$

We see that $f'(x)$ is odd.

(c) For $n > 0$, calculate

$$\int_0^\pi f(x)\cos nx\,dx = f(x)\frac{\sin nx}{n}\Big|_0^\pi - \int_0^\pi f'(x)\frac{\sin nx}{n}\,dx = -\frac{1}{n}\int_0^\pi f'(x)\sin nx\,dx.$$

Note a_0 cannot be found by integrating the derivative, because the constant is lost in differentiating. But for $n = 1, 2, 3, \ldots$,

$$a_n = -\frac{1}{n}\frac{2}{\pi}\int_0^\pi f'(x)\sin nx\,dx.$$

(d) So here is the point. Because $f(x)$ is even, $f'(x)$ is odd. But odd functions have Fourier sine series, so

$$f'(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_0^\pi f'(x)\sin n\,dx,$$

and the calculation in part **(c)** shows that

$$a_n = -\frac{1}{n}b_n.$$

Suppose the Fourier sine series for $f'(x)$ converges. (Just for emphasis, this means that *the coefficients b_n decay at a rate sufficient to make the sum converge for every x.*) But a_n is b_n divided by n. Thus the a_n decay faster than the b_n. Thus the Fourier cosine series for $f(x)$ converges. Faster. In fact it always converges to a smoother function; $f(x)$ is smoother than $f'(x)$.

We see the relationship between a less-smooth derivative and a more-smooth function in lots of examples. For instance, the derivative may have jumps and Gibbs effect, but the function itself has no jumps and better convergence at the sharp corners than at the jumps of the derivative.