Final Exam

(Total of 150 points.)

Due Thursday December 16, 2004 at NOON *FIRM*.

Put it into my box in the department office or under my door.

1. (*15 pts*) Find the Fourier transform of

$$f(x) = xe^{-x^2}.$$

In particular, compute the integrals necessary to find the Fourier transform. [*Undoubtedly the answer can be looked up in some book. But I want the completely elementary calculus technique which, along with a calculation done in class, gives the answer.*]

2. (*15 pts*) Read Lesson 43 in the textbook. Explain in your own words what the "tour du wino" is, what PDE problem it solves, and (to the extent you can explain this) why it solves this problem. Use at most *six* sentences for your explanation. Now state—no need to rederive—the analytical solution to this PDE problem. (*Hint.* Do this problem *after* you have done another particular problem on this final.)

3. (*10 pts*) Find all possible solutions $u = u(x, y, z)$ to the PDE

$$u_{xz} = 0.$$

4. (*15 pts*) Solve the heat equation problem

$$\begin{array}{rc} \text{PDE} & u_t = 5u_{xx} \\ \text{BCs} & u_x(-1,t) = 0 \\ & u_x(1,t) = 0 \\ \text{IC} & u(x,0) = \sin(\pi x/2) \end{array}$$

for $u = u(x,t)$, $-1 \le x \le 1$ and $t \ge 0$. Graph the solution at $t = 0, 0.1, 0.2, 0.3, 0.4$.

Suggest an amount of time which is a "natural time scale" for this problem. [*There is no perfect answer for the last part, but there is probably a small range of good answers. In any case, the last part ("*Suggest *...") will not be worth many points.*]

5. (*15 pts*) Lesson 8, # 3.

6. (*20 pts*) Solve the following PDE boundary value problem using separation-of-variables and Fourier series:

$$\begin{array}{rll} \text{PDE} & u_{xx} + u_{yy} = 0 & 0 < x < 1,\ 0 < y < 1; \\ \text{BCs} & u(x,1) = 1, & 0 < x < 1, \\ & u(x,0) = 0, & 0 < x < 1, \\ & u(0,y) = 0, & 0 < y < 1, \\ & u(1,y) = 0, & 0 < y < 1; \end{array}$$

[*Be sure to check your answer for reasonable-ness. In particular, attempt to graph the first or second partial sum and also look at sums of your series when you restrict to the boundary.*]

7. (*15 pts*) Lesson 9, # 4.

8. (*15 pts*) Suppose $f = f(x)$ is a function with period 2π which equals $x^2 + \sin x$ on the interval $-\pi < x \leq \pi$. Find the Fourier series of f. Then determine, with justification from the appropriate theorem, those points x at which the Fourier series converges to $f(x)$.

9. (*10 pts*) Solve the following wave equation problem:

$$\begin{array}{rl} \text{PDE} & u_{tt} = 5u_{xx}, \quad -\infty < x < \infty,\, t \geq 0; \\ \text{ICs} & u(x,0) = e^{-x^2}, \quad -\infty < x < \infty, \\ & u_t(x,0) = 0, \quad -\infty < x < \infty. \end{array}$$

Graph the solution at times $t = 0, 0.2, 0.4$. [*Hint.* Easy. *Produce good by-hand graphs.*]

10. (*20 pts*) [*Perhaps you should work on this problem last. I will give major partial credit for substantial work toward the goal.*] Solve the Dirichlet problem on the interior of the sphere for $u = u(r, \theta, \phi)$:

$$\nabla^2 u = 0, \qquad 0 < r < 1,\, 0 \leq \theta \leq 2\pi,\, 0 \leq \phi \leq \pi,$$

$$u(1, \theta, \phi) = \begin{cases} 1, & 0 \leq \phi < \pi/2, \\ -1, & \pi/2 < \phi \leq \pi. \end{cases}$$

Use spherical coordinates r, θ, ϕ as in Lesson 35.

Rules for this Take–Home Exam

1. Ed Bueler is the only person with whom you may communicate regarding this exam.
2. You may use any *book* (published before December 1, 2004) you wish, but *you must give references* for books other than the textbook.
3. You may work on this final in the great outdoors.
4. I will try to detect the use of *Google*, etc., as a source of solutions to problems. And I will punish it or ask the Dean of Students to do so.
5. Do not put more than one problem per page (i.e. at most two problems per sheet).
6. I will deduct points for unreadable messes!
7. You may work on this final wearing whatever clothing you want, but I don't want to know about it.
8. When you turn in your exam, sort your solutions into the order given and *staple* in the upper left corner.
9. Write your name on your exam when you turn it in.