Selected Solutions to Assignment #9

Total of 12 points.

Lesson 8, # 1: *Not graded.* **Answer in the back of the text.**

Lesson 8, # 2: (*6 pts*) As suggested, we transform the problem twice. First,

$$u(x,t) = x + U(x,t)$$

defines a new function U which satisfies homogeneous boundary conditions:

$$\begin{array}{rl} \text{PDE} & U_t = U_{xx} - U \\ \text{BCs} & \begin{array}{l} U(0,t) = 0 \\ U(1,t) = 0 \end{array} \\ \text{IC} & U(x,0) = -x \end{array}$$

This PDE is handled by the first transformation in lesson 8,

$$U(x,t) = e^{-t} w(x,t),$$

which gives the following system for w:

$$\begin{array}{rl} \text{PDE} & w_t = w_{xx} \\ \text{BCs} & \begin{array}{l} w(0,t) = 0 \\ w(1,t) = 0 \end{array} \\ \text{IC} & w(x,0) = -x \end{array}$$

This problem is familiar. The PDE and BCs give

$$w(x,t) = \sum_{j=1}^{\infty} a_j e^{-(j\pi)^2 t} \sin(j\pi x).$$

The constants must be chosen so that

$$-x = w(x,0) = \sum_{j=1}^{\infty} a_j \sin(j\pi x).$$

We can do the necessary integrals but we can also look it up. In appendix 1, table D we find

$$x = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin(nx)$$

for x *in the interval* $[0,\pi]$. Substitute $x = \pi\tilde{x}$ and replace n with j to get

$$\tilde{x} = \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{2}{j} (-1)^{j+1} \sin(j\pi\tilde{x})$$

for $\tilde{x}$ in the interval $[0,1]$; this is what we want.

It follows that $a_j = -\frac{2}{\pi j}(-1)^{j+1}$ and

$$w(x,t) = -\frac{2}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} e^{-(j\pi)^2 t} \sin(j\pi x)$$

so
$$u(x,t) = x - \frac{2}{\pi} e^{-t} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} e^{-(j\pi)^2 t} \sin(j\pi x).$$
(In terms of checking this solution, note that $\overline{u}(x) = \lim_{t\to\infty} u(x,t)$ satisfies the obvious steady state ODE with BCs, and that $u(x,0) = x - x = 0$ as desired.)

Lesson 9, # 2: (*6 pts*) First we note that the homogeneous version of the boundary value problem, namely
$$\begin{array}{cl} \text{PDE} & u_t = u_{xx} \\ \text{BCs} & \begin{array}{l} u(0,t) = 0 \\ u(1,t) = 0 \end{array} \end{array}$$
has $X_j(x) = \sin(j\pi x)$ as its eigenfunctions.

Thus it is natural to seek a solution of the form
$$u(x,t) = \sum_{j=1}^{\infty} T_j(t) \sin(j\pi x)$$
to the full problem. Substitution into the PDE gives
$$\sum_{j=1}^{\infty} T_j'(t) \sin(j\pi x) = \sum_{j=1}^{\infty} -(j\pi)^2 T_j(t) \sin(j\pi x) + \sin(\pi x) + \sin(2\pi x).$$
Also the initial condition gives
$$\sum_{j=1}^{\infty} T_j(0) \sin(j\pi x) = 0.$$
The idea is that these equalities show equality of the coefficients (of the orthogonal functions "$\sin(j\pi x)$").

Thus we have a sequence of ODE problems:
$$\begin{aligned} T_1'(t) &= -\pi^2 T_1(t) + 1, & T_1(0) &= 0, \\ T_2'(t) &= -2^2\pi^2 T_2(t) + 1, & T_2(0) &= 0, \\ T_3'(t) &= -3^2\pi^2 T_3(t), & T_3(0) &= 0, \\ T_4'(t) &= -4^2\pi^2 T_4(t), & T_4(0) &= 0, \\ T_5'(t) &= -5^2\pi^2 T_5(t), & T_5(0) &= 0, \\ &\vdots \end{aligned}$$
Only the first two have nonzero solutions:
$$\begin{aligned} T_1(t) &= \frac{1}{\pi^2}\left(1 - e^{-\pi^2 t}\right), \\ T_2(t) &= \frac{1}{4\pi^2}\left(1 - e^{-4\pi^2 t}\right). \end{aligned}$$
Thus
$$u(x,t) = \frac{1}{\pi^2}\left(1 - e^{-\pi^2 t}\right)\sin(\pi x) + \frac{1}{4\pi^2}\left(1 - e^{-4\pi^2 t}\right)\sin(2\pi x).$$