Selected Solutions to Assignment #8

Total of 18 points.

Exercise I: (*6 pts*) I give only the high points

SEPARATION-OF-VARIABLES.

$$u(x,t) = X(x)T(t) \text{ and PDE} \qquad \Longrightarrow \qquad \frac{c^2T''}{T} = -\lambda^2 = \frac{X''}{X}.$$

EIGENVALUE PROBLEM AND ITS SOLUTION.

$$X'' + \lambda^2 X = 0,\ X(0) = 0,\ X(L) = 0 \qquad \Longrightarrow \qquad \lambda_j = \frac{j\pi}{L}, \quad X_j(x) = \sin(j\pi x/L).$$

$T(t)$ PROBLEMS AND ITS SOLUTION.

$$c^2T'' + \lambda^2 T = 0 \qquad \Longrightarrow \qquad T(t) = c_1\cos(\lambda t/c) + c_2 \sin(\lambda t/c).$$

EXPANSION.

$$u(x,t) = \sum_{j=1}^\infty \left[a_j\cos(\lambda_j t/c) + b_j \sin(\lambda_j t/c)\right]\sin(\lambda_j x).$$

ORTHOGONALITY.

$$\int_0^L \sin(\lambda_j x)\sin(\lambda_k x)\,dx = \begin{cases} 0, & j \ne k, \\ \frac{L}{2}, & j = k. \end{cases}$$

HOMOGENEOUS INITIAL CONDITION.

$$0 = u(x,0) = \sum_{j=1}^\infty \left[a_j \cdot 1 + b_j \cdot 0\right]\sin(\lambda_j x) \qquad \Longrightarrow \qquad a_j = 0.$$

NON-HOMOGENEOUS INITIAL CONDITION.

$$g(x) = u_t(x,0) = \sum_{j=1}^\infty b_j \frac{j\pi}{cL}\sin(\lambda_j x) \qquad \Longrightarrow \qquad \boxed{b_j = \frac{2c}{j\pi}\int_0^L g(x)\sin\left(\frac{j\pi x}{L}\right)\,dx}.$$

FINAL ANSWER.

$$\boxed{u(x,t) = \sum_{j=1}^\infty b_j \sin\left(\frac{j\pi t}{cL}\right)\sin\left(\frac{j\pi x}{L}\right)}$$

along with the boxed formula for b_j.

Exercise II: (*Not graded.*)

SEPARATION-OF-VARIABLES.

$$u(x,t) = X(x)T(t) \text{ and PDE} \qquad \Longrightarrow \qquad \frac{T''}{T} = -\lambda^2 = \frac{X''}{X}.$$

EIGENVALUE PROBLEM AND ITS SOLUTION.

$$X''+\lambda^2 X = 0,\ X(0) = 0,\ X'(1) = 0 \qquad \Longrightarrow \qquad \lambda_j = (2j-1)\pi,\ X_j(x) = \sin(\lambda_j x), j = 1,2,3,\dots.$$

$T(t)$ PROBLEMS AND ITS SOLUTION.

$$T'' + \lambda^2 T = 0 \qquad \Longrightarrow \qquad T(t) = c_1\cos(\lambda t) + c_2\sin(\lambda t).$$

Expansion.

$$u(x,t) = \sum_{j=1}^{\infty} [a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t)] \sin(\lambda_j x).$$

Orthogonality.

$$\int_0^1 \sin(\lambda_j x) \sin(\lambda_k x)\, dx = \begin{cases} 0, & j \neq k, \\ \frac{1}{2}, & j = k. \end{cases}$$

Homogeneous initial condition.

$$0 = u_t(x,0) = \sum_{j=1}^{\infty} [-a_j \lambda_j \cdot 0 + b_j \lambda_j \cdot 1] \sin(\lambda_j x) \qquad \Longrightarrow \qquad b_j = 0.$$

Non-homogeneous initial condition.

$$5\cos((\pi/2)x) = u(x,0) = \sum_{j=1}^{\infty} a_j \sin(\lambda_j x) \qquad \Longrightarrow \qquad a_j = 10 \int_0^1 \cos((\pi/2)x) \sin((2j{-}1)\pi x/2)\, dx.$$

Actual integral computation. Using $\cos a \sin b = \frac{1}{2}\left(\sin(a+b) - \sin(a-b)\right)$,

$$\begin{aligned} \int_0^1 \cos(\pi x/2) \sin((2j-1)\pi x/2)\, dx &= \frac{1}{2} \int_0^1 \sin(2j\frac{\pi}{2}x) - \sin(2(1-j)\frac{\pi}{2}x)\, dx \\ &= \frac{1}{2} \int_0^1 \sin(j\pi x) + \sin((j-1)\pi x)\, dx \\ &= \frac{1}{\pi} \begin{cases} (j-1)^{-1}, & j \text{ odd}, \\ j^{-1}, & j \text{ even}. \end{cases} \end{aligned}$$

Final answer.

$$u(x,t) = \frac{10}{\pi} \sum_{j=1}^{\infty} \eta_j \cos((2j-1)\pi t/2) \sin((2j-1)\pi x/2)$$

where $\eta_j = j^{-1}$ if j is even and $\eta_j = (j-1)^{-1}$ if j is odd.

The question was intended to be simpler, with $u(x,0) = 5\sin(\pi x/2)$. *With the actual initial condition, the request* "Sketch $u(x,t)$ at several times" *is impractical, as well as the request to give a physical interpretation.*

Lesson 7, # 1: (*Not graded.*) **Answer in back of text.**

Lesson 7, # 2: (*3 pts*) First, $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$ in the text's form of the Sturm-Liouville problem. Second, a standard computation gives

$$\lambda_j = ((2j-1)\pi/2)^2, \qquad X_n(x) = \sin((2j-1)\pi x/2), \qquad j = 1, 2, 3, \ldots$$

(Note: *The text uses "λ" where I tend to write "λ^2" in class. The solution here matches the problem statement in the text, of course.*)

Lesson 7, # 3: (*6 pts*) [*Recommendation: read solution to #4 first.*]

Separation-of-variables.

$$u(x,t) = X(x)T(t) \text{ and PDE} \qquad \Longrightarrow \qquad \frac{T'}{T} = -\lambda = \frac{X''}{X}.$$

Eigenvalue problem and its solution.

$X''+\lambda X = 0,\ X'(0) = 0,\ X'(1) = 0 \qquad \Longrightarrow \qquad \lambda_j = (j\pi)^2,\ X_j(x) = \cos(j\pi x), j = 0, 1, 2, 3, \dots.$

$T(t)$ problems and its solution.

$$T' + \lambda T = 0 \qquad \Longrightarrow \qquad T(t) = e^{-\lambda t}.$$

Expansion.

$$u(x,t) = a_0 + \sum_{j=1}^{\infty} a_j e^{-(j\pi)^2 t} \cos(j\pi x).$$

Orthogonality.

$$\int_0^1 \cos(j\pi x)\cos(k\pi x)\,dx = \begin{cases} 0, & j \neq k, \\ 1, & j = k = 0, \\ \frac{1}{2}, & j = k > 0. \end{cases}$$

Non-homogeneous initial condition.

$$x = u(x,0) = a_0 + \sum_{j=1}^{\infty} a_j \cos(j\pi x) \qquad \Longrightarrow \qquad a_j = \begin{cases} \int_0^1 x\,dx, & j = 0, \\ 2\int_0^1 x\cos(j\pi x)\,dx \end{cases}$$

$$= \begin{cases} \frac{1}{2}, & j = 0, \\ 0, & j > 0 \text{ even}, \\ -\frac{4}{(j\pi)^2}, & j > 0 \text{ odd}. \end{cases}$$

Final answer.

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{j=1}^{\infty} \frac{e^{-((2j-1)\pi)^2 t}\sin((2j-1)\pi x)}{(2j-1)^2}.$$

[*Note*: The solution in the back is in error in that it does not have zero coefficients when the index is even.]

Lesson 7, # 4: (*3 pts*) Here we should start with considering cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. The boundary conditions do not allow nontrivial solutions if $\lambda > 0$. For $\lambda = 0$,

$$X(x) = c_1 x + c_2 \text{ \& BCs} \qquad \Longrightarrow \qquad X(x) = 1,$$

while for $\lambda < 0$,

$X(x) = c_1\cos(\sqrt{\lambda}x)+c_2\sin(\sqrt{\lambda}x)$ & BCs $\quad \Longrightarrow \quad \lambda_j = (j\pi)^2,\ X_j(x) = \cos(j\pi x),\ j = 1, 2, 3, \dots$

Thus the eigenvalues are $\lambda_j = (j\pi)^2$, $j = 0, 1, 2, \dots$, and $X_j(x) = \cos(j\pi x)$, $j = 0, 1, 2, \dots$