

**Exam # 2 SOLUTIONS.**

**1 (a).**  $\mathbf{v}(t) = 2t\mathbf{i} + \cos(\pi t)\mathbf{j} + \mathbf{k}$ ,  
 $\mathbf{a}(t) = 2\mathbf{i} - \pi \sin(\pi t)\mathbf{j}$ .

**1 (b).** At  $t = 1$ ,  $\mathbf{v}(1) = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ ,  $\mathbf{a}(1) = 2\mathbf{i}$ . Thus

$$\theta = \cos^{-1} \left( \frac{\mathbf{v}(1) \cdot \mathbf{a}(1)}{|\mathbf{v}(1)||\mathbf{a}(1)|} \right) = \cos^{-1} \left( \frac{4}{\sqrt{6} \cdot 2} \right) = \cos^{-1} \left( \frac{2}{\sqrt{6}} \right).$$

**2.** (a)  $\leftrightarrow$  II; (b)  $\leftrightarrow$  I; (c)  $\leftrightarrow$  III; (d)  $\leftrightarrow$  IV.

**3 (a).** First,  $\mathbf{v} = -\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}$  so  $|\mathbf{v}| = \sqrt{5}$ . Thus  $s(t) = \int_0^t |\mathbf{v}(\tau)| d\tau = \int_0^t \sqrt{5} d\tau = \sqrt{5}t$ .

**3 (b).** We need  $\mathbf{T}$  and  $|d\mathbf{T}/dt|$ . Since  $|\mathbf{v}| = \sqrt{5}$  it follows that  $\mathbf{T} = \frac{1}{\sqrt{5}}(-\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k})$ . Thus

$$\frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{5}}(-\cos t\mathbf{i} - \sin t\mathbf{j});$$

$$\kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{\sqrt{5}} \frac{1}{\sqrt{5}} = \frac{1}{5}.$$

3 (c).

4.

5. By the fundamental theorem of calculus I,

$$\frac{\partial f}{\partial y} = e^{-y^2}; \quad \frac{\partial f}{\partial x} = -e^{-x^2}.$$

6. We need the direction of  $\mathbf{v}$  and also the gradient of  $f$ . Then we can take the directional derivative:

$$\begin{aligned} \hat{\mathbf{u}} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{3}{5}, 0, -\frac{4}{5} \right\rangle; \\ \nabla f &= \mathbf{i} + 3y^2\mathbf{j} + 2z\mathbf{k}; \quad \nabla f|_{(0,1,2)} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}; \\ D_{\hat{\mathbf{u}}}f &= \nabla f|_{(0,1,2)} \cdot \hat{\mathbf{u}} = -\frac{13}{5}. \end{aligned}$$

[Note that the answer is a scalar not a vector!]

7. Here  $f(x, y, z) = x^2 + y^2 + z^2$  and the surface is a level surface of  $f$ . The gradient will be the normal vector to the plane we want:

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}; \quad \mathbf{n} = \nabla f|_{(1,1,-1)} = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}.$$

The plane is  $\mathbf{n} \cdot \mathbf{PP}_0 = 0$ :

$$2(x - 1) + 2(y - 1) - 2(z + 1) = 0$$

(or  $x + y - z = 3$ ).

8. The chain rule is

$$\frac{dw}{dt} = \frac{\partial w}{\partial u} \frac{du}{dt} + \frac{\partial w}{\partial v} \frac{dv}{dt}$$

and the diagram is at right:

**Extra Credit.** In this example we can do without calculus. In fact, the curve  $\mathbf{r}(t)$  can be drawn onto the level curves of  $f$  and you will see the answer. Note that the curve is  $x = t, y = t^2 + 1$  or equivalently  $y = x^2 + 1$ . Then we see from the level curves of  $f$  that the minimum is at the point  $(x = 0, y = 1)$ , which happens when  $t = 1$ , and the value of  $f$  is 1. See the picture for 4 above.

Of course, calculus can also be used!