Exam # 1 Solutions.

1. This is just a question of normalizing. The two answers are

$$\hat{\mathbf{u}} = (3/5)\mathbf{i} - (4/5)\mathbf{j} \text{ and } -\hat{\mathbf{u}} = -(3/5)\mathbf{i} + (4/5)\mathbf{j}.$$

2. Implicit derivative first:

$$6x + 8y + 8x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$$
$$\iff \qquad \frac{dy}{dx} = \frac{-6x - 8y}{8x + 4y}.$$

Thus $m =$ (slope at $(1,0)$) is $m = -6/8 = -3/4$. A normal vector to a line of this slope is $\mathbf{n} = 3\mathbf{i} + 4\mathbf{j}$. Thus the unit normals are

$$\hat{\mathbf{n}} = (3/5)\mathbf{i} + (4/5)\mathbf{j} \text{ and } -\hat{\mathbf{n}} = -(3/5)\mathbf{i} - (4/5)\mathbf{j}.$$

3. I claim the area is $A_{tri} = \frac{1}{2}|\mathbf{PQ} \times \mathbf{PR}|$ since the magnitude of the cross product is the area of the enclosed parallelogram. Thus we calculate:

$$\mathbf{PQ} = \langle 1, -1, -3 \rangle; \qquad \mathbf{PR} = \langle -1, 1, -1 \rangle;$$
$$\mathbf{PQ} \times \mathbf{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -3 \\ -1 & 1 & -1 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j};$$

and conclude:

$$A_{tri} = (1/2)\sqrt{16 + 16} = 2\sqrt{2}.$$

4. We need a normal vector $\mathbf{n}$ to the plane and we build that as $\mathbf{n} = \mathbf{PQ} \times \mathbf{PR}$, but we did this calculation above:

$$\mathbf{n} = \mathbf{PQ} \times \mathbf{PR} = 4\mathbf{i} + 4\mathbf{j}.$$

Thus the plane is $\mathbf{n} \cdot \mathbf{P}_{(x,y,z)}\mathbf{P}_0 = 0$ where $\mathbf{P}_0 = \mathbf{P}$, or in other words

$$4(x-1) + 4(y-1) + 0(z-2) = 0,$$
$$\text{or} \qquad 4x + 4y = 8 \quad \text{or} \quad x + y = 2.$$

5. Integrate twice and include the info to give "**c**":

$$\frac{d\mathbf{r}}{dt} = -32t\mathbf{j} + \mathbf{c} \qquad \text{and } \mathbf{c} = 8\mathbf{i} + \mathbf{j};$$
$$\mathbf{r}(t) = 8t\mathbf{i} + (-16t^2 + t)\mathbf{j} + \mathbf{c}_1 \qquad \text{and } \mathbf{c}_1 = 100\mathbf{i};$$

so

$$\mathbf{r}(t) = (8t + 100)\mathbf{i} + (-16t^2 + t)\mathbf{j}.$$

[*It's easy to check that this is right!*]

6.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ -1 & -2 & 1 \end{vmatrix} = -2\mathbf{i}+\mathbf{j}.$$

The answer to the question at the bottom of the page is "NEITHER" because if $\mathbf{u}$ and $\mathbf{v}$ were parallel then the above cross product would have given $\mathbf{0}$; on the other hand, if the two vectors were perpendicular then the dot product would be zero, but it is not:

$$\mathbf{u}\cdot\mathbf{v} = -1-4-2 = -7.$$

[*This question was worth* 2 *of the* 10 *points. I understand that many did not see it, but there were only three lines of text on the page when you got the exam. Be a bit alert, please. Extra credit questions may also appear at the bottoms of pages in the future.*]

7. It is a circular hoop [*or "loop" or "ring" or just "circle", but not "disk", which is solid*] parallel to [*not "in"*] the x, z plane. It lies in the $y = 2$ plane, has radius 1 and is centered at $(0, 2, 0)$.

8. Recall

$$\operatorname{proj}_{\mathbf{v}}\mathbf{F} = \left(\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

by definition so

$$\begin{aligned}\mathbf{F} &= (\operatorname{proj}_{\mathbf{v}}\mathbf{F}) + (\mathbf{F}-\operatorname{proj}_{\mathbf{v}}\mathbf{F}) \\ &= \left((\frac{300}{3})(\mathbf{i}+\mathbf{j}+\mathbf{k})\right) + (\ldots) \\ &= (100\mathbf{i}+100\mathbf{j}+100\mathbf{k}) + (100\mathbf{i}+200\mathbf{j}-(100\mathbf{i}+100\mathbf{j}+100\mathbf{k})) \\ &= (100\mathbf{i}+100\mathbf{j}+100\mathbf{k}) + (100\mathbf{j}-100\mathbf{k}).\end{aligned}$$

The first vector is parallel to $\mathbf{v}$ and the second is perpendicular, and they do indeed add up to $\mathbf{F}$. [*Check these facts!*]

9. One way (along the lines of chapter 10) is to find $\mathbf{v}(0)$ and $\mathbf{r}(0)$ and then the parametric/vector equation for the line is $\mathbf{l}(t) = \mathbf{r}(0)+\mathbf{v}(0)t$:

$$\begin{aligned} &\mathbf{r}(0) = 0\mathbf{i}+3\mathbf{j}; \\ \mathbf{v}(t) = \cos t\,\mathbf{i}+(-\sin t)\mathbf{j}; \quad &\Longrightarrow \quad \mathbf{v}(0) = 1\mathbf{i}+0\mathbf{j}; \\ \Longrightarrow \quad &\mathbf{l}(t) = t\mathbf{i}+3\mathbf{j}.\end{aligned}$$

This is the line $y = 3$ of course.

Also, the problem could be done by using the vector $\mathbf{v}(0)$ to get a slope and $\mathbf{r}(0)$ to get a point and then using the point–slope form of the line.

Extra Credit. This has to do with *angles* and I only gave more than one point for explicitly addressing the angle.

One way to start is to calculate:

$$\begin{aligned}\theta(t) &= (\text{angle between the } x\text{–axis and the tangent vector } \mathbf{v}(t)) \\ &= \arccos\left(\frac{\mathbf{i}\cdot\mathbf{v}(t)}{1|\mathbf{v}(t)|}\right) = \arccos\left(\frac{\cos t}{1}\right) \\ &= t.\end{aligned}$$

Thus the *rate* at which this angle is changing is

$$\frac{d\theta}{dt} = 1\frac{\text{rad}}{(\text{time unit})}.$$