

### Midterm Exam # 2 Solutions

1. Recall  $S = 4\pi r^2$ . We want  $dS/dr$ :

$$\frac{dS}{dr} = 8\pi r \quad \text{so when } r = 3 \text{ we get} \quad \frac{dS}{dr} = 24\pi.$$

2. We know  $dV/dt = 36$ . We want  $dr/dt$ . Recall

$$V = \frac{4}{3}\pi r^3.$$

Thus

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

which is rewritten as

$$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} = \frac{36}{4\pi \cdot 9} = \frac{1 \text{ cm}}{\pi \text{ sec}}$$

when  $r = 3$  cm.

3. (a)

$$\frac{dy}{dx} = \frac{1 \cdot \cos x - x(-\sin x)}{\cos^2 x} = \frac{\cos x + x \sin x}{\cos^2 x}$$

(b)

$$f'(t) = \frac{1}{3}(1 + \tan t)^{-2/3} \sec^2 t = \frac{\sec^2 t}{3(1 + \tan t)^{2/3}}$$

(c)

$$g'(x) = (\ln 3)3^{(x^2)} \cdot 2x = (2 \ln 3) x 3^{(x^2)}$$

4. (a)

$$3x^2 + 2xy + x^2 \frac{dy}{dx} + 8y \frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx} = \frac{-3x^2 - 2xy}{x^2 + 8y}.$$

(b) The slope is

$$m = \left. \frac{dy}{dx} \right|_{(1,1)} = \frac{-3 - 2}{1 + 8} = -\frac{5}{9}$$

and the equation of the tangent line is

$$y - 1 = (-5/9)(x - 1).$$

5. The function is differentiable for all  $x$ . The derivative is

$$f'(x) = \frac{1 \cdot (x^2 + 1) - x \cdot (2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

The critical points are where  $f'(x) = 0$ , that is,

$$1 - x^2 = 0 \quad \text{or} \quad x = \pm 1.$$

The only critical point of interest is  $x = 1$  because  $x = -1$  is outside of  $[0, 2]$ . We make a table of  $f$  values, and include the endpoints:

$x$	$f(x)$
0	0
1	$\frac{1}{1+1} = \frac{1}{2}$
2	$\frac{2}{2^2+1} = \frac{2}{5}$

Thus the absolute maximum is at  $x = 1$  with value  $f(1) = 1/2$  while the absolute minimum is at  $x = 0$  with value  $f(0) = 0$ .

6. [A sketch will be given in class.]

7. (a)

$$\frac{d}{dx}(\sec x) = \frac{d}{dx} \left( \frac{1}{\cos x} \right) = \frac{0 \cdot \cos x - 1 \cdot (-\sin x)}{\cos^2 x} = \frac{1 \cdot \sin x}{\cos x \cdot \cos x} = \sec x \tan x.$$

(b) Rewrite  $y = \sec^{-1} x$  as  $\sec y = x$  and do implicit differentiation:

$$\sec y \tan y \frac{dy}{dx} = 1.$$

Thus

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y \sqrt{\sec^2 y - 1}} = \frac{1}{x \sqrt{x^2 - 1}}$$

because  $\tan^2 y + 1 = \sec^2 y$  and  $\sec y = x$ .

8. (a) Here  $f'(x) = 1/x$  so

$$L(x) = f(a) + f'(a)(x - a) = (\ln 1) + \frac{1}{1}(x - 1) = 0 + (x - 1) = x - 1.$$

(b) [A sketch will be given in class.]

(c) **Extra Credit** We have

$$\ln 4 = f(4) \approx L(4) = 4 - 1 = 3.$$

This is not a very good approximation because  $x = 4$  is too far from  $a = 1$ . In fact,  $1 = \ln e^1 < \ln 4 < \ln e^2 = 2$  because  $e < 4 < e^2$ . So  $1 < \text{error} = |3 - \ln 4| < 2$ .

9. **The Mean Value Theorem.** In fact Rolle's Theorem also works. See exercise #35 in section 4.2. Note that the Intermediate Value Theorem and the Extreme Value Theorem have nothing to do with derivatives, and thus nothing to do with *speed*.