Midterm Exam # 1 Solutions

1. *Something like*: "The values of $f(x)$ can be made to be as close as one wishes to L by choosing x sufficiently close to, but not equal to, a."

2. (a)

$$\begin{aligned}(f \circ g)(x) &= \sin\left(\frac{1}{1+x}\right),\\ (f \circ f)(x) &= \sin(\sin x),\\ (g \circ f)(x) &= \frac{1}{1+\sin x}.\end{aligned}$$

(b)

$$\begin{aligned}&\text{domain } f \circ g \text{ is } \{x \neq 1\},\\ &\text{domain } f \circ f \text{ is } \{\text{all real numbers}\},\\ &\text{domain } g \circ f \text{ is } \{\sin x \neq 1\} = \left\{ x \text{ is not one of } \dots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \dots \right\}.\end{aligned}$$

3. (a) $f'(x) = -2x + 6x^5 - 18x^{17}$.
(b)

$$g'(x) = \frac{2x(x-1) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}.$$

4. The original limit gives the indeterminant form "$\infty - \infty$". Thus we complete the difference of squares:

$$\begin{aligned}\lim_{x\to\infty} \sqrt{x^2+1} - x &= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\left(\sqrt{x^2+1} + x\right)} = \lim_{x\to\infty} \frac{(x^2+1) - x^2}{\left(\sqrt{x^2+1} + x\right)}\\ &= \lim_{x\to\infty} \frac{1}{\left(\sqrt{x^2+1} + x\right)} = 0.\end{aligned}$$

5. [*Sketch will be given in class.*]

6.

$$\begin{aligned}\lim_{h\to 0} \frac{(1+h)^4 - 1}{h} &= \lim_{h\to 0} \frac{1 + 4h + 6h^2 + 4h^3 + h^4 - 1}{h} = \lim_{h\to 0} \frac{h(4 + 6h + 4h^2 + h^3)}{h}\\ &= \lim_{h\to 0} 4 + 6h + 4h^2 + h^3 = 4 + 0 + 0 + 0 = 4.\end{aligned}$$

7. [*Sketch will be given in class.*]

8. [*Sketch will be given in class.*]

9. First,

$$\frac{dy}{dx} = 3 + (-1)x^{-2} = 3 - \frac{1}{x^2}.$$

[*Note that you may use the power rule to calculate the derivative. The definition of the derivative also works, of course, but if I want you to use it I will definitely say "Use the definition of the derivative to ..."*]

Thus

$$m = \left.\frac{dy}{dx}\right|_{x=1} = 3 - 1 = 2$$

so the tangent line is

$$y - 4 = 2(x - 1) \qquad \text{or} \qquad y = 2x + 2$$

because we know a point $(x_0, y_0) = (1, 4)$.

Extra Credit. The point here is that as one approaches the origin from either left or right on the graph $y = x^{2/3}$, the slope goes to infinity. [*It is not enough to note that the slope is undefined at $x = 0$, because there are plenty of graphs like $y = |x|$ where the slope is undefined at $x = 0$ but no one would say "the tangent line is vertical."*] The calculations which support my claim that the slope goes to infinity are:

$$m_{\text{tangent,right}} = \lim_{h \to 0^+} \frac{h^{2/3} - 0}{h} = \lim_{h \to 0^+} h^{-1/3} = +\infty,$$

$$m_{\text{tangent,left}} = \lim_{h \to 0^-} \frac{h^{2/3} - 0}{h} = \lim_{h \to 0^-} h^{-1/3} = -\infty,$$

so

$$|m_{\text{tangent,right}}| = |m_{\text{tangent,left}}| = +\infty$$

so the tangent line at the origin is vertical (even though the derivative at $x = 0$ does not exist).