Math 200 Review: April 8, 2008

1. Find the derivative:

   (a) $y = \left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)^2$

   Solution: $y' = 2\left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)\left(\frac{1}{2\sqrt{x}} - \frac{1}{3\sqrt[3]{x^4}}\right)$

   (b) $y = 4\pi^2$

   Solution: $y' = 0$

   (c) $y = ae^{x^2} + \frac{b}{x} + \frac{c}{x^2}$

   Solution: $y' = 2axe^{x^2} - \frac{b}{x^2} - \frac{2c}{x^3}$

   (d) $y = \frac{v^3 - 2v\sqrt{v}}{v}$

   Solution: $y' = 2v - \frac{1}{\sqrt{v}}$

   (e) $x^3y^2 = x^2 + \sin y \tan x$

   Solution: $\frac{dy}{dx} = \frac{x^2 - 3x^2y^2 + \sin y \sec x}{2x^3y - \cos y \tan x}$

   (f) $y = \arctan(\arcsin\sqrt{x})$

   Solution: $y' = \left(\frac{1}{1 + (\arcsin\sqrt{x})^2}\right)\left(\frac{1}{\sqrt{1-x}}\right)\left(\frac{1}{2\sqrt{x}}\right)$

   (g) $y = \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}$

   Solution: $y' = \frac{8x(x^2+1)^3(2x+1)^3(3x-1)^5 - 6(2x+1)^2(3x-1)^5(x^2+1)^4 - 10(3x-1)^4(2x+1)^3(x^2+1)^4}{((2x+1)^3(3x-1)^5)^2}$

2. Prove that $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

   Solution: First rewrite in terms of $\sin$ and $\cos$, then

$$\begin{aligned}\frac{d}{dx}(\csc x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{0 - \cos x}{(\sin x)^2} \\ &= -\frac{1}{\sin x}\frac{\cos x}{\sin x} = -\csc x \cot x.\end{aligned}$$

3. Find an equation of the tangent line to the curve at the given point

   (a) $y = 4\sin^2 x$ at $(\frac{\pi}{6}, 1)$
   Solution: First we must find the derivative $y' = 4\sin 2x$, by the double angle formula. Then we plug in the point to get the slope. So, $y' = m = 2\sqrt{3}$. Then using the point-slope formula we get $y = 2\sqrt{3}x - \frac{\pi}{3}\sqrt{3} + 1$.

   (b) $y = \dfrac{x^2 - 1}{x^2 + 1}$ at $(0, -1)$
   Solution: Following the same process we get $y' = \frac{2x(x^2+1)-(x^2-1)2x}{(x^2+1)^2}$ then $y' = m = 0$. So the equation for tangent line is $y = 1$.

4. A bacteria culture contains 200 cells initially and grows at a rate proportional to its size. After half an hour the population has increased to 360 cells.

   (a) Find the number of bacteria after t hours.
   Solution: Using the formula $y(t) = y(0)e^{kt}$ we find that $y(0) = 200$ and $k = 1.18$ because
   $$360 = 200e^{k(1/2)}$$
   or
   $$k = 2\ln\left(\frac{360}{200}\right) = 2\ln\left(\frac{9}{5}\right) = 1.17557\ldots$$
   So the number of bacteria after t hours is $y(t) = 200e^{1.18t}$.

   (b) Find the number of bacteria after 4 hours.
   Solution: $y(4) = 22040$ cells

   (c) Find the rate of growth after 4 hours.
   Solution: $\dfrac{dy}{dt} = 236e^{1.18t}$ so when $t = 4$ then $\dfrac{dy}{dt} = 26472$ cells per hour

   (d) When does the population reach $10,000$ cells.
   Solution: $10000 = 200e^{1.18t} \Rightarrow t = 3.3$hours

5. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?

   Solution: Note that the volume of a cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius and h is the height. Also note that $\dfrac{r}{h} = \dfrac{3}{10} \Rightarrow r = \dfrac{3}{10}h$. So $V = \frac{3\pi}{100}h^3$. The rate of change of the volume is 2 cm^3/s. We want the rate of change of height:
   $$\frac{dV}{dt} = \frac{9\pi}{100}h^2\frac{dh}{dt} \qquad \Rightarrow \qquad \frac{dh}{dt} = \frac{8}{9\pi}\frac{\text{cm}}{\text{s}}.$$

6. Find the absolute maximum and absolute minimum values of the functions on the given intervals:

   (a) $f(x) = x^3 - 3x + 1$ on $[0, 3]$

   Solution: $f'(x) = 3x^2 - 3$, so $0 = 3x^2 - 3 \Rightarrow x = \pm 1$. Note $x = -1$ is not in the interval on which we are maximizing/minimizing. Checking the end points of the interval along with the one critical number in the interval, I get

   $$f(0) = 1, \qquad f(1) = -1, \qquad f(3) = 19$$

   The absolute maximum is when $x = 3$ where $f(3) = 19$ and the absolute minimum is when $x = 1$ where $f(1) = -1$.

   (b) $f(x) = \dfrac{x}{x^2 + 1}$ on $[0, 2]$

   Solution: $f'(x) = \frac{-x^2+1}{x^2+1}$, so $0 = -x^2 + 1 \Rightarrow x = \pm 1$. Again $x = -1$ is not in the interval of interest. Checking all the other points I get

   $$f(0) = 0, \qquad f(1) = \frac{1}{2}, \qquad f(2) = \frac{2}{5}$$

   Then the absolute maximum is when $x = 2$ where $f(1) = \frac{1}{2}$ and the absolute minimum is when $x = 0$ where $f(0) = 0$.