

Ph.D. Comprehensive Exam Syllabus: Algebra

Authors: Allman/Rhodes (Oct. 4, 2007)

Groups:

- basics of finite and infinite group theory
- composition series
- group actions and Sylow Theorems

Rings and Modules:

- basic ring theory concepts and examples (including PIDs, UFDs, Euclidean domains, multivariable polynomial rings, matrix rings)
- ideals in commutative rings (prime, maximal, radical of ideals, ideal quotients $I:J$, nilradical, $\text{Spec}(A)$)
- construction of finite fields $\mathbb{F}(p^n)$.
- local rings and localizations
- modules, $\text{Hom}_A(M, N)$
- exact sequences
- integral extensions and closure
- Noetherian and Artinian rings/modules, chain conditions
- Hilbert Basis Theorem and Nullstellensatz
- Nakayama's Lemma
- primary decomposition

Abstract Vector Spaces:

- basic theory over a field
- dual spaces
- Cayley-Hamilton Theorem
- Jordan canonical form
- tensor product of vector spaces

Galois Theory:

- finite algebraic extensions of \mathbb{Q}
- field automorphisms
- splitting fields
- separable/inseparable extensions
- the Galois correspondence
- application to non-solvability of the general quintic

Comprehensive references for all topics:

- Dummit and Foote, *Abstract Algebra*, 3rd ed., Chapters 1-5, 7-15
- Hungerford, *Algebra*, Chapter I-VIII

Supplementary references:

- Artin, *Galois Theory: Lectures Delivered at the University of Notre Dame*
- Atiyah and MacDonald, *Introduction to Commutative Algebra*
- Herstein, *Topics in Algebra*
- Jacobson, *Lectures in Abstract Algebra I,II,III*
- Kaplansky, *Commutative Rings*
- Stewart, *Galois Theory*, 3rd ed.