1. Carothers 1.29

2. Determine if the following functions are continuous.

a) $F : (C[0,1], L_1) \to \mathbb{R}$, $F(f) = f(1)$

b) $G : (C[0,1], L_2) \to (C[0,1], L_1)$, $G(f) = f$

c) $H : (C[0,1], L_1) \to (C[0,1], L_2)$, $H(f) = f$

You are welcome to use the Cauchy-Schwartz inequality for integrals.

3. Prove that ℓ_2 is separable.

4. For each interval $[-n, n]$, define $d_n : C[-n, n] \times C[-n, n] \to \mathbb{R}$ by

$$d_n(f, g) = \frac{||f - g||_\infty}{1 + ||f - g||_\infty}.$$

Recall from your work on Carothers 3.42 that d_n is a metric on $C[-n, n]$ that is equivalent to the metric induced by the L_∞ norm.

Define $d : C(\mathbb{R}) \times C(\mathbb{R})$ by

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} d_n(f, g).$$

(Here we are identifying f, g with their restrictions to each $[-n, n]$).

a) Prove that d is a metric on $C(\mathbb{R})$.

b) Prove that $f_n \to g$ with respect to d if and only if $f_n \rightrightarrows g$ on every compact set $K \subseteq \mathbb{R}$.

5. Let $f \in C(\mathbb{R})$, and let $f_n(x) = f(x + 1/n)$. Prove or disprove: $f_n \to f$ in $C(\mathbb{R})$ (with respect to the norm introduced in the previous problem).

6. Carothers 8.53

7. Recall that a step function on $[a, b]$ is a function $g : [a, b] \to \mathbb{R}$ such that there exists a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that g is constant on each open interval (x_{k-1}, x_k).

a) Give a necessary and sufficient condition for a step function to be lower semicontinuous. You do not need to be rigorous here.

b) Suppose $f : [a, b] \to \mathbb{R}$. Show that f is lower-semicontinuous if and only if there is an increasing sequence (g_n) of lower-semicontinuous step functions converging pointwise to f.

8. Let $\mathcal{X}$ be the space of closed subsets of a compact space X, and let H be Hausdorff distance on $\mathcal{X}$. Let (F_n) be a Cauchy sequence in $\mathcal{X}$, and define

$$D_n = \overline{\bigcup_{k \geq n} F_k}.$$

a) Prove that $\lim_{n\to\infty} H(F_n, D_n) = 0$.

b) Prove that (F_n) converges. Conclude that $\mathcal{X}$ is complete.

c) Prove that $\mathcal{X}$ is totally bounded. Conclude that $\mathcal{X}$ is compact.

9. Carothers 11.11

Rules and format:

- You are welcome to discuss this exam with me (David Maxwell) to ask for hints and so forth.
- If you find a suspected typo, please contact me as soon as possible and I will communicate it to the class if needed.
- You may not discuss the exam with anyone else until after the due date/time.
- You are permitted to reference Carothers or any other real analysis text you like. If you use another text, you must cite it when you use it.
- Each problem is weighted equally.
- The due date/time is absolutely firm.
- The problem session on October 30 will be a hints session. We may schedule another if there is sufficient demand.