1. Abbott 2.4.2

a) Prove that the sequence defined by $x_1 = 3$ and
$$x_{n+1} = \frac{1}{4 - x_n}$$
converges.

b) Now that we know that $\lim_n x_n$ exists, explain why $\lim_n x_{n+1}$ exists.

c) Take the limit of each side of the recursive equation and explicitly compute $\lim x_n$.

Solution, part a:
We first show that the sequence is bounded between $2 + \sqrt{3}$ and $2 - \sqrt{3}$. Certainly this is true for x_1. Suppose for some k, $2 - \sqrt{3} < x_k < 2 + \sqrt{3}$. Then
$$2 + \sqrt{3} > 4 - x_k > 2 - \sqrt{3}$$
and
$$\frac{1}{2 + \sqrt{3}} < \frac{1}{4 - x_k} < \frac{1}{2 - \sqrt{3}}.$$
Notice,
$$\frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3} \quad \text{and} \quad \frac{1}{2 - \sqrt{3}} = 2 + \sqrt{3}.$$
Hence $2 - \sqrt{3} < x_{k+1} < 2 + \sqrt{3}$. So $2 - \sqrt{3} < x_n < 2 + \sqrt{3}$ for every $n \in \mathbb{N}$.
Now notice that if $2 - \sqrt{3} < x < 2 + \sqrt{3}$, then
$$x^2 - 4x + 1 = (x - (2 - \sqrt{3}))(x - (2 + \sqrt{3}) < 0.$$
So
$$x > \frac{1}{4 - x}.$$
In particular, for every n,
$$x_n > \frac{1}{4 - x_n} = x_{n+1}$$
and the sequence is monotone decreasing.

Solution, part b:
Since (x_{n+1}) is a subsequence of the convergent sequence (x_n), it has the same limit.

Solution, part c:
Let $x = \lim_n x_n$. Taking the limit of both sides of the equation results in
$$x = \frac{1}{4 - x}.$$
Hence x is a root of the quadratic $x^2 - 4x + 1$, so $x = 2 \pm \sqrt{3}$. But $x \leq 3$, so $x = 2 - \sqrt{3}$.

2. Abbott 2.4.4 Show that

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converges and find the limit.

Solution:
Let $a_1 = \sqrt{2}$ and define $a_{k+1} = \sqrt{2a_k}$ for $k \geq 1$. We claim this sequence is monotone increasing and bounded above by 2. Certainly $a_1 \leq 2$. And if $a_k \leq 2$, then $a_{k+1}^2 = 2a_k \leq 4 = 2^2$. So $a_k \leq 2$ for every k.

Now for any k, $2a_k \geq a_k a_k$, and hence $a_{k+1} = \sqrt{2a_k} \geq \sqrt{(a_k)^2} = a_k$. Hence the sequence is monotone increasing and bounded above by 2. But then the sequence converges to some number a. Now $a_{n+1}^2 = 2a_n$. Since $\lim a_{n+1}^2 = a^2$ and since $\lim 2a_n = 2a$ we have

$$a^2 = 2a.$$

So $a = 0$ or $a = 2$. Since the sequence is monotone increasing and $a_1 > 0$, we conclude that $a = 2$.

3.

a) Let $x, y > 0$ and $k \in \mathbb{N}$. Prove that $x > y$ if and only if $x^k > y^k$. *Hint:* The proof is by induction.

b) Let $x, y > 0$ and $k \in \mathbb{N}$. Prove that $x > y$ if and only if $x^{1/k} > y^{1/k}$. *Hint:* Use part *a*.

c) Let $c > 1$. Prove that $c^{1/(n+1)} \leq c^{1/n}$ for every n. *Hint:* Compare $\left[c^{1/(n+1)}\right]^{(n+1)}$ and $\left[c^{1/n}\right]^{(n+1)}$.

d) Compute (with proof) $\lim_{n\to\infty} c^{1/n}$ if $c > 1$.

Solution, part a:
The case $k = 1$ is obvious. So suppose for some k that $x > y$ if and only if $x^n > y^k$. Then, if $x > y$, then $x^{n+1} = x^n x > y^n x > y^n y = y^{n+1}$. Similarly, if $x \leq y$, then $x^{n+1} \leq y^{n+1}$. So if $x^{n+1} > y^{n+1}$, the we must have $x > y$.

Solution, part b:
Suppose $x > y$. Then $(x^{1/k})^k > (y^{1/k})^k$, so $x^{1/k} > y^{1/k}$ by part a. Suppose $x^{1/k} > y^{1/k}$. Then again by part *a*,

$$x = (x^{1/k})^k > (y^{1/k})^k = y.$$

Solution, part c:
Let $c > 1$. Note that $c^{1/n} > 1^{1/n} = 1$. Also,

$$(c^{1/n})^{n+1} = c^{1+1/n} = cc^{1/n} > c.$$

Hence $(c^{1/(n+1)})^{n+1} = c < (c^{1/n})^{n+1}$. By part *b* we conclude that $c^{1/(n+1)} < c^{1/n}$.

Solution, part d:
We have already noted that $c^{1/n}$ is a decreasing sequence and is bounded below by 1. So it converges to a limit L. But then the subsequence $b_n = c^{1/(2n)}$ also converges to L. However,

$$L^2 = \lim_{n\to\infty} b_n^2 = \lim_{n\to\infty} (c^{1/(2n)})^2 = \lim_{n\to\infty} c^{1/n} = L.$$

Hence $L = 0$ or $L = 1$. Since $c^{1/n} \geq 1$ for every n, the Limit Order Theorem implies $L \geq 1$> So $L = 1$.

4. Abbott 2.5.1

Solution:
Let (a_n) be a convergent sequence converging to a, and let (a_{n_k}) be a subsequence. Let $\epsilon > 0$. Pick N so that if $n \geq N$, then $|a_n - a| < \epsilon$. An easy proof by induction shows that $n_k \geq k$ for every k. Now let $K = N$. Notice that if $k \geq K$, then $n_k \geq k \geq K = N$. Hence $|a_{n_k} - a| < \epsilon$. Hence (a_{n_k}) converges to a.

5. Let (a_n) be an unbounded sequence. Prove there is a subsequence (a_{n_k}) such that $\lim_k 1/a_{n_k} = 0$.

Solution:
Pick n_1 so that $|a_{n_1}| \geq 1$. This is possible since the sequence is unbounded. For each $k > 1$, pick n_k such that $n_k > n_{k-1}$ and $|a_{n_k}| \geq k$. This is possible, for otherwise if $|a_n| < k$ for every $n > n_k$, then $|a_n| \leq \max(|a_1|, |a_2|, \ldots, |a_{n_k}|, k)$ for every n and the sequence would be bounded.

Hence we have constructed a subsequence (a_{n_k}) such that $|a_{n_k}| \geq k$ for every k. Now let $b_k = 1/a_{n_k}$. Note that $0 < b_k \leq 1/k$ for every k. The Squeeze theorem then implies $\lim b_k = 0$. That is, $\lim 1/a_{n_k} = 0$.

6. Abbott 2.5.2

a) Assume $a_1 + a_2 + \cdots$ is a convergent series converging to L. Show that any regrouping

$$(a_1 + a_2 + \cdots + a_{n_1}) + (a_{n+1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) + \cdots$$

leads to a series that also converges to L.

b) Explain why this proof does not contradict the example in section 1 showing that infinite addition is not associative.

Solution, part a:
Let $s_n = a_1 + \cdots + a_n$. That the series $\sum a_n$ converges to L is exactly the statement that $\lim s_n = L$. let $n_0 = 0$, so we can write the regrouped sum as the series

$$\sum_{k=0}^{\infty} (a_{n_k+1} + \cdots + a_{n_k}).$$

Notice that the partial sums t_k of this series are exactly the partial sums s_{n_k}. Since $\lim s_{n_k} = L$, we have $\lim t_k = L$ also.

Solution, part b:
This result does not apply to the last example from section 1, since the original series diverged.

7. Abbott 2.5.4 Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit a. Prove that $\lim a_n = a$.

Solution:
Suppose to the contrary that (a_n) does not converge to a. Then there exists an $\epsilon > 0$ such that for any $N \in \mathbb{N}$, there exists $n \geq N$ such that $|a - a_n| \geq \epsilon$.

We construct a subsequence as follows. Pick $n_1 \geq 1$ such that $|a - a_{n_1}| \geq \epsilon$. Now pick $n_2 > n_1$ (i.e. $n_2 \geq n_1 + 1$) such that $|a - a_{n_2}| \geq \epsilon$. This is possible since $a_n \not\to a$. Continuing in this fashion, we can construct a subsequence (a_{n_k}) such that $|a_{n_k} - a| \geq \epsilon$.

Let $b_k = a_{n_k}$. This is again a bounded sequence, and hence has a convergent subsequence, $(b_{k_r})_r$. A subsequence of a subsequence is a subsequence of the original sequence. So by hypothesis, $\lim_r b_{k_r} = a$. But $|b_{k_r} - a| \geq \epsilon$ for every r. This is a contradiction.