1. Stillwell 5.3.3 Suppose that A, B, C, and D are four points such that no three are colinear. Prove that if AB and BC have a common point E, then $E = B$.

Solution:
Suppose to the contrary that $E \neq B$. Then by Axiom 1 there is a unique line EB passing through E and B. But AB is a line passing through E and B. So $EB = AB$. Similarly, $EB = BC$. Hence $AB = BC$ and A, B, and C are colinear. This is a contradiction.

2. Stillwell 5.3.4 Deduce that AB, BC and CD do not have a common point, nor do any three of AB, BC, CD, and DA.

Solution:
Suppose to the contrary that E is a point on AB, BC, and CD. By 5.3.3, $E = B$. By an identical argument to 5.3.3, we also know that $E = C$. Hence $B = C$. But this is a contradiction since the points A, B, C, and D are distinct.

3. Stillwell 5.4.1 Find the plane $ax + by + cz = 0$ that contains the points $(1, 2, 3)$ and $(1, 1, 1)$.

Solution:
Interpreting $ax + by + cz = 0$ as $(a, b, c) \cdot (x, y, z) = 0$ we need to find a vector (a, b, c) such that it is perpendicular to both of $(1, 2, 3)$ and $(1, 1, 1)$. We can find such a vector by computing a cross product, so

$$(a, b, c) = (1, 2, 3) \times (1, 1, 1) = (-1, 2, -1).$$

The plane is then

$$-x + 2y - z = 0$$

and it is easy to verify that this plane indeed contains the two points.

4. Stillwell 5.4.2 Find the line formed by the intersection of the planes $x + 2y + 3z = 0$ and $x + y + z = 0$.

Solution:
A point (a, b, c) on the line must satisfy

$$(1, 2, 3) \cdot (a, b, c) = 0$$

and

$$(1, 1, 1) \cdot (a, b, c) = 0.$$

So (a, b, c) must be perpendicular to both of $(1, 1, 1)$ and $(1, 2, 3)$. We already found one such vector in the last problem, namely $(-1, 2, -1)$. Hence the line is given by

$$\lambda(-1, 2, -1)$$

where $\lambda \in \mathbb{R}$.

5. 5.4.3 We have the principle of duality for projective planes. The plane $ax + by + cz = 0$ contains the line spanned by $t(\alpha, \beta, \gamma)$ (for $t \in \mathbb{R}$) if and only if the plane $\alpha x + \beta y + \gamma c = 0$ contains the line spanned by $t(a, b, c)$ (for $t \in \mathbb{R}$).

Hence the plane $ax + by + cz$ contains the points $(1, 1, 1)$ and $(1, 2, 3)$ if and only if the planes $x + y + z = 0$ and $x + 2y + 3z = 0$ contain the point (a, b, c) and the line spanned by $t(a, b, c)$. Since an answer to 5.4.1 was $(-1, 2, -1)$, an answer to 5.4.2 must also be $(-1, 2, -2)$.