1. Consider the function $f(x) = x/(x^3+16)$. Find the critical points of f and use the First Derivative Test to determine if each critical point is a local minimum, local maximum, or neither.

Solution:
Taking the derivative of f gives us:

$$f'(x) = \frac{(x^3+16) - x(3x^2)}{(x^3+16)^2} = \frac{-2x^3+16}{(x^3+16)^2}.$$

The critical points are the points x where $f(x)$ equals zero or is undefined. This will occur where $-2x^3+16 = 0$, or where $x^3+16 = 0$. So the critical points are $x = 2, -2\sqrt{2}$

Applying the first derivative test, around $x = -2\sqrt{2}$, we see that $f'(x) > 0$. At $x = \sqrt{2}$, we see $f'(x)$ goes from being positive to negative. So $x = \sqrt{2}$ is a local maximum, and $x = -2\sqrt{x}$ is neither a local maximum or minimum.

2. Consider the function $f(t) = t^2 + 1/t + 1$. Find the critical points of f and use the Second Derivative Test to determine if each critical point is a local minimum, local maximum, or neither. Also, find the points of inflection of $f(t)$.

Solution:
Taking the derivative of f gives us:

$$f'(t) = \frac{2t(t+1) - (t^2+1)(1)}{(t+1)^2} = \frac{t^2+2t-1}{(t+1)^2}.$$

The critical points are the points t where $t^2+2t-1 = 0$ and $t+1 = 0$. So the critical points are $t = -1-\sqrt{2}, -1$, and $-1+\sqrt{2}$

We next find the second derivative of f:

$$\begin{aligned} f''(t) &= \frac{(2t+2)(t+1)^2 - (t^2+2t-1)[2(t+1)]}{(t+1)^4} \\ &= \frac{(2t+2)(t+1) - 2(t^2+2t-1)}{(t+1)^3} \\ &= \frac{2t^2+4t+2-2t^2-4t+2}{(t+1)^3} \\ &= \frac{4}{(t+1)^3}. \end{aligned}$$

We can see that $f''(-1+\sqrt{2}) > 0$, so f has a local minimum at $t = -1+\sqrt{2}$, and $f''(-1-\sqrt{2}) < 0$, so f has a local maximum at $t = -1-\sqrt{2}$. While $f''(-1)$ is undefined, we do see that f changes concavity at that point, since $f''(t) < 0$ for $t < -1$ and $f''(t) > 0$ for $t > -1$.

3. Consider the function $f(x) = e^{-x^2}$.

a) Where is $f(x)$ positive? Negative? Equal to zero?

Solution:
$f(x) > 0$ for all values of x.

b) What is $f(0)$?

Solution:
$f(0) = e^0 = 1$.

c) On what intervals is f increasing? Decreasing?

Solution:
Taking the derivative of f gives us

$$f'(x) = -2xe^{-x^2}.$$

So $f'(x) > 0$, and therefore $f(x)$ is increasing, where $x < 0$. $f'(x) < 0$, and therefore $f(x)$ is decreasing, where $x > 0$.

d) What are the critical points of $f(x)$?

Solution:
The critical points of $f(x)$ occur where $f'(x) = 0$, or where $x = 0$.

e) On what intervals is f concave up? Concave down?

Solution:
Taking the derivative of $f'(x)$ gives us:

$$f''(x) = 4x^2e^{-x^2} - 2e^{-x^2} = (4x^2 - 2)e^{-x^2}.$$

So the function will be concave down where $4x^2 - 2 < 0$, or on the interval $(-1/\sqrt{2}, 1/\sqrt{2})$. The function will be concave up where $4x^2 - 2 > 0$, or on the intervals $(-\infty, -1/\sqrt{2})$ and $(1/\sqrt{2}, \infty)$.

f) What are the points of inflection of f?

Solution:
The inflection points of f occur where $f''(x) = 0$, or at $x = \pm 1/\sqrt{2}$.

g) What is $\lim_{x\to\infty} e^{-x^2}$?

Solution:
Notice that

$$f(x) = e^{-x^2} = \frac{1}{e^{x^2}}.$$

Since $e^{x^2} \to \infty$ as $x \to \infty$, then $\lim_{x\to\infty} f(x) = 0$.

h) What is $\lim_{x\to-\infty} e^{-x^2}$?

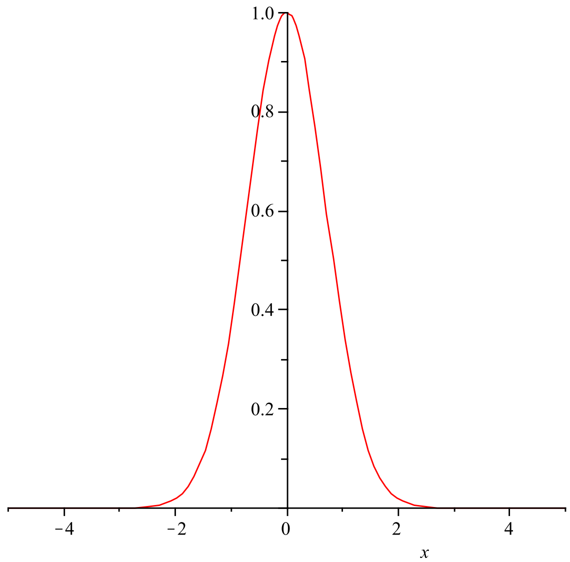
Solution:
Notice

$$f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x).$$

So this function is even. Therefore, $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0$.

i) Use the wealth of information you have found to sketch the graph of $f(x)$.

Solution:



4. Consider the function $f(x) = x^5 - 5x$.

a) Where is $f(x)$ positive? Negative? Equal to zero?

Solution:
Notice

$$f(x) = x(x^4 - 5).$$

From this, we can see $f(x) = 0$ where $x = 0, -\sqrt[4]{5}, \sqrt[4]{5}$. We see $f(x) < 0$ on $(-\infty, -\sqrt[4]{5})$ and $(0, \sqrt[4]{5})$, and $f(x) > 0$ on $(-\sqrt[4]{5}, 0)$ and $(\sqrt[4]{5}, \infty)$.

b) What is $f(0)$?

Solution:
As we just showed, $f(0) = 0$.

c) On what intervals is f increasing? Decreasing?

Solution:
Differentiating f gives us

$$f'(x) = 5x^4 - 5.$$

The function f is increasing where $f'(x) > 0$, or when $x^4 > 1$. This will occur on the intervals $(-\infty, -1)$ and $(1, \infty)$. The function f is decreasing where $f'(x) < 0$, or on the interval $(-1, 1)$.

d) What are the critical points of $f(x)$?

Solution:
The critical points of f occur where $f'(x) = 0$, or where $x = -1, 1$.

e) On what intervals is f concave up? Concave down?

Solution:
Differentiating $f'(x)$ gives us

$$f''(x) = 20x^3.$$

We know f is concave up where $f''(x) > 0$, or where $x > 0$. f is concave down, where $f''(x) < 0$, or where $x < 0$.

f) What are the points of inflection of f?

Solution:
The inflections points of f occur where $f''(x) = 0$, or where $x = 0$.

g) What is $\lim_{x\to\infty} x^5 - 5x$?

Solution:
$\lim_{x\to\infty} x^5 - 5x = \infty$.

h) What is $\lim_{x\to-\infty} x^5 - 5x$?

Solution:
Notice

$$f(-x) = (-x)^5 - 5(-x) = -(x^5 - 5x) = -f(x).$$

So f is an odd function. Thus, $\lim_{x\to-\infty} f(x) = -\lim_{x\to\infty} f(x) = -\infty$.

i) Use the wealth of information you have found to sketch the graph of $f(x)$.

Solution: