1. Let $f(x) = x + \sqrt{5 + x^2}$.

a) What is the linearization $L(x)$ of f at the point $x = 2$?

Solution: $L(x) = f(2) + f'(2)(x - 2)$. Notice

$$f'(x) = 1 + \frac{x}{\sqrt{5 + x^2}}$$

Thus,

$$\begin{aligned} L(x) &= [2 + \sqrt{5+4}] + \left[1 + \frac{2}{\sqrt{5+4}}\right](x-2) \\ &= 5 + \left(\frac{5}{3}\right)(x-2) \\ &= \frac{5}{3} + \frac{5}{3}x \end{aligned}$$

b) Show that $L(2) = f(2)$ and $L'(2) = f'(2)$.

Solution:

$$f(2) = 2 + \sqrt{5+4} = 5 = \frac{5}{3} + \frac{5}{3}(2) = L(2)$$

$$f'(2) = 1 + \frac{2}{\sqrt{5+4}} = \frac{5}{3} = L'(2)$$

c) Compute $f(x)$ and $L(x)$ for $x = 3$, $x = 2.5$, $x = 2.1$, and $x = 2.01$.

Solution:

$$f(3) \approx 6.7417 \qquad f(2.5) \approx 5.8541 \qquad f(2.1) \approx 5.1676 \qquad f(2.01) \approx 5.0167$$

$$L(3) \approx 6.6667 \qquad L(2.5) \approx 5.8333 \qquad L(2.1) \approx 5.1667 \qquad L(2.01) \approx 5.0167.$$

d) Find a quadratic polynomial $Q(x)$ such that $Q(2) = f(2)$, $Q'(2) = f'(2)$, and $Q''(2) = f''(2)$.

Solution: Notice

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left[1 + \frac{x}{\sqrt{5+x^2}}\right] = \frac{1}{\sqrt{5+x^2}} + x\left(-\frac{1}{2}(5+x^2)^{-3/2}2x\right) \\ &= \frac{1}{\sqrt{5+x^2}} - \frac{x^2}{(5+x^2)^{3/2}} \end{aligned}$$

From this, we get

$$f''(2) = \frac{1}{3} - \frac{4}{27} = \frac{5}{27}$$

With this in mind, we can find $Q(x)$. We already know $L(x)$ and $L'(x)$ approximate $f(x)$ and $f'(x)$, respectively, for values of x near $x = 2$. Ideally, to find

$Q(x)$, we add a term onto $L(x)$ such that $Q(2) = L(2)$ and $Q'(2) = L(2)$, but that $Q''(2) = f''(2)$. This leads us to $Q(x) = L(x) + \frac{f''(2)}{2}(x-2)^2$. Note the multiplication by $1/2$ here, which is done to offset the effect of the power rule when we differentiate. Since we just found $f''(x)$, we can simplify $Q(x)$ as follows:

$$\begin{aligned} Q(x) &= L(x) + \frac{f''(2)}{2}(x-2)^2 \\ &= \frac{5}{3} + \frac{5x}{3} + \left(\frac{5}{27}\right)(x^2 - 2x + 4) \\ &= \frac{5}{3} + \frac{5x}{3} + \frac{5x^2}{27} - \frac{10x}{27} + \frac{20}{27} \\ &= \frac{65}{27} + \frac{35x}{27} + \frac{5x^2}{27}. \end{aligned}$$

Notice

$$Q(2) = f(2) + f'(2)(2-2) + \frac{f''(2)}{2}(2-2)^2 = f(2).$$

Differentiating Q gives us

$$Q'(x) = f'(2) + f''(2)(x-2).$$

So $Q'(2) = f'(2)$. Differentiating Q' gives us

$$Q''(2) = f''(2).$$

Therefore $Q(2) = f(2)$, $Q'(2) = f'(2)$, and $Q''(2) = f''(2)$, as desired.

e) Compute $Q(x)$ for $x = 3$, $x = 2.5$, $x = 2.1$, and $x = 2.01$.

Solution:

$Q(3) \approx 7.9730$ $\quad$ $Q(2.5) \approx 6.8056$ $\quad$ $Q(2.1) \approx 5.9463$ $\quad$ $Q(2.01) \approx 5.7611$

f) Conjecture what a good strategy might be for finding a cubic polynomial that approximates $f(x)$ near $x = 2$.

Solution: as we did with $Q(x)$, we will find our desired cubic polynomial, which we will call $C(x)$ by adding on a term so that $C(x) = Q(x)$, $C'(x) = Q'(x)$, $C''(x) = Q''(x)$ and $C'''(x) = f'''(x)$. From this, we get:

$$C(x) = Q(x) + \frac{f'''(a)}{6}(x-a)^3.$$

2. A cylindrical storage tank is known to be 10m high. It has an interior radius of r meters and an exterior radius of R meters.

a) About how accurately should r be measured to calculate the tank's volume to within 1% of its true value? Your answer will depend on r.

Solution: The volume of the interior of the tank is given by $V = 10\pi r^2$. In order to find the relative error, we first differentiate, and find the differential:

$$\frac{dV}{dr} = 20\pi r$$
$$dV = 20\pi r dr.$$

Since we want to get the relative error to be less than 1%, we want the ratio of dV to the volume, $\frac{dV}{V}$ to be less than $.01$. Notice

$$\frac{dV}{V} = \frac{20\pi r dr}{10\pi r^2} = \frac{2}{r}dr$$

So if $\frac{dV}{V} < .01$, then $\frac{2}{r}dr < .01$, or $dr < .005r$. Therefor, in order to calculate the volume to within 1% of its true value, we need to measure r to with $.5\%$ of its true value.

b) About how accurately should R be measured to calculate the amount of paint it will take to paint the side of the tank to within .5% of its true value? Your answer will depend on R.

Solution: The surface area of the exterior of the tank is given by $S = 20\pi R$. As before, we differentiate to find the differential:

$$\frac{dS}{dR} = 20\pi$$
$$dS = 20\pi dR.$$

We want $\frac{dS}{S} < .05$. Notice

$$\frac{dS}{S} = \frac{20\pi dR}{20\pi R} = \frac{dR}{R}$$

Therefore, if $\frac{dS}{S} < .05$, then $\frac{dR}{R} < .05$, or $dR = .05R$. Therefore, in order to calculate S to within 5% of its true value, we need to measure R to within 5% of its true value.