

1. Compute

$$\lim_{x \rightarrow 0^+} x \ln(x).$$

Solution:

Notice:

$$\lim_{x \rightarrow 0^+} x \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x}$$

which we can see goes to $\frac{-\infty}{\infty}$ as $x \rightarrow 0$ from the right. Applying L'Hospital's rule, we get:

$$\lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} -x = 0.$$

2. Compute

$$\lim_{x \rightarrow 0^+} x^{\sin(x)}.$$

Solution:Let $y = x^{\sin(x)}$. Then

$$\ln(y) = \sin(x) \ln(x) = \frac{\sin(x)}{1/\ln(x)}$$

We see from this that

$$\lim_{x \rightarrow 0^+} \ln(y) = \lim_{x \rightarrow 0^+} \frac{\sin(x)}{1/\ln(x)}$$

which goes to $\frac{0}{0}$ as $x \rightarrow 0$ from the right. Applying L'Hospital's rule, we get:

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln(y) &= \lim_{x \rightarrow 0^+} \frac{\cos(x)}{1/x} \\ &= \lim_{x \rightarrow 0^+} x \cos(x) \\ &= 0. \end{aligned}$$

So $\lim_{x \rightarrow 0^+} \ln(y) = 0$, and so $\lim_{x \rightarrow 0^+} y = 1$.

3. Suppose a wire of length 10cm is cut into two pieces, one of which is bent into a square and the other is curled into a circle. What length should the two pieces have in order to minimized the sum of the area of the square and the circle?

Solution:

Suppose the length of wire used for the circle is of length x . Then the length of wire used to make the square is of length $10 - x$. So we have a square with a circumference of x . From the formula for circumference we get

$$r = \frac{x}{2\pi}.$$

So the area of this circle is $\pi r^2 = \frac{x^2}{4\pi}$. We also have a square with a perimeter of $10 - x$, and so this square has area $[\frac{1}{4}(10 - x)]^2$.

We want to find the value of x that minimizes the function

$$A(x) = \frac{x^2}{4\pi} + \left[\frac{1}{4}(10 - x) \right]^2 = \frac{x^2}{4\pi} + \frac{1}{16}(100 - 20x + x^2) = \frac{x^2}{4\pi} + \frac{25}{4} - \frac{5x}{4} + \frac{x^2}{16}.$$

We now take the derivative of A and setting it equal to zero gives us:

$$A'(x) = \frac{x}{2\pi} - \frac{5}{4} + \frac{x}{8} = 0.$$

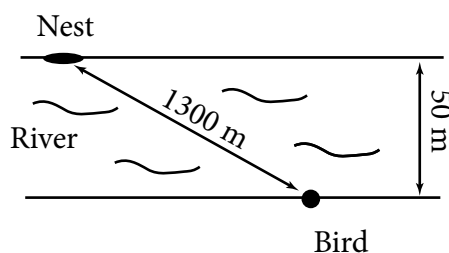
This simplifies to:

$$\frac{4 + \pi}{8\pi}x - \frac{5}{4} = 0.$$

Which gives us $x = \frac{10\pi}{4+\pi}$. We can also see that A' goes from being positive to negative at this point. So $x = \frac{10\pi}{4+\pi}$ is a minimum of this function. Thus, the area is minimized if a length of $\frac{10\pi}{4+\pi} \approx 4.399$ cm is used for the circle and $10 - \frac{10\pi}{4+\pi} \approx 5.601$ is used for the square.

4. Homing pigeons fly faster over land than over water. Assume they fly 10 meters per second over land but only 8 meters per second over water.

If a pigeon is located at the edge of a straight river 50 meters wide and must fly to its nest located 1300 meters away on the opposite side of the river, what path minimizes its flying time?



Solution:

The Pythagorean Theorem tells us that the pigeon is roughly 1299 meters of the point on the river directly across from the bank. Suppose the pigeon flies $1299 - x$ meters along the bank parallel to the river, and then flies from there directly to its nest over the water, a distance of $\sqrt{50^2 + x^2}$. Recall that $\text{time} = \frac{\text{distance}}{\text{rate}}$.

All this gives us a function for time:

$$T(x) = \frac{1}{10}(1299 - x) + \frac{1}{8}\sqrt{50^2 + x^2}.$$

Differentiating T and setting $T' = 0$ gives us:

$$T'(x) = -\frac{1}{10} + \frac{2x}{8\sqrt{50^2 + x^2}} = 0.$$

This can be simplified to

$$20x = 8\sqrt{50^2 + x^2}.$$

Squaring both sides gives us:

$$400x^2 = 64(50^2) + 64x^2$$

This simplifies to:

$$336x^2 = 160,000$$

Thus, we get $x = \pm\sqrt{\frac{160,000}{336}} \approx \pm 21.82$ m. Ignoring the negative answer, we get $x \approx 21.82$ m.

So the pigeon's flight time is minimized if it flies $1299 - x \approx 1277.18$ meters along the bank, and the flies $\sqrt{50^2 + x^2} \approx 54.55$ m over the water.