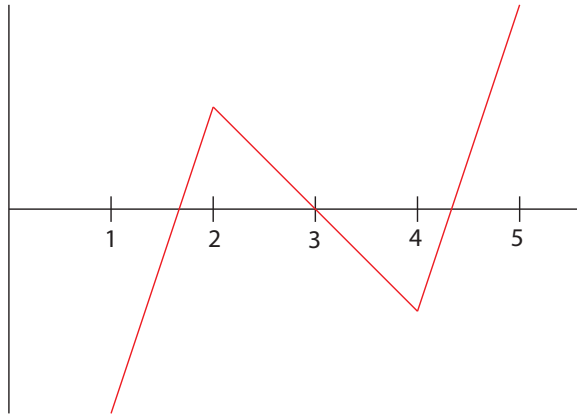


CALCULUS HOMEWORK SOLUTIONS
WEEK 9

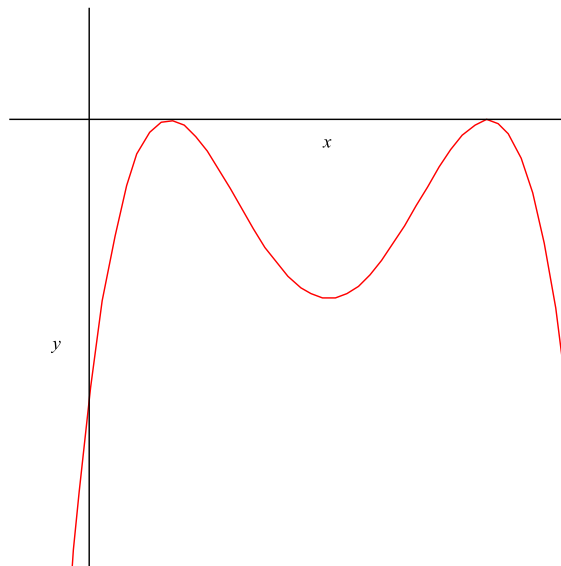
§4.1, 8: Sketch the graph of a function f that is continuous on $[1, 5]$, has an absolute minimum at 1, absolute maximum at 5, local maximum at 2, and local minimum at 4.

Solution:



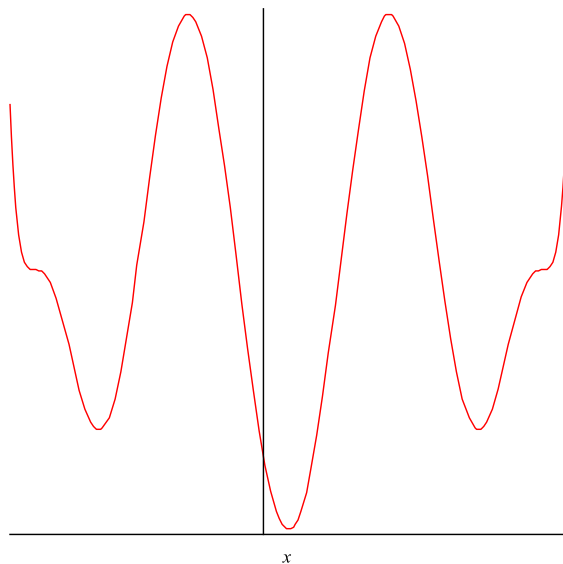
§4.1, 14a: Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum

Solution:



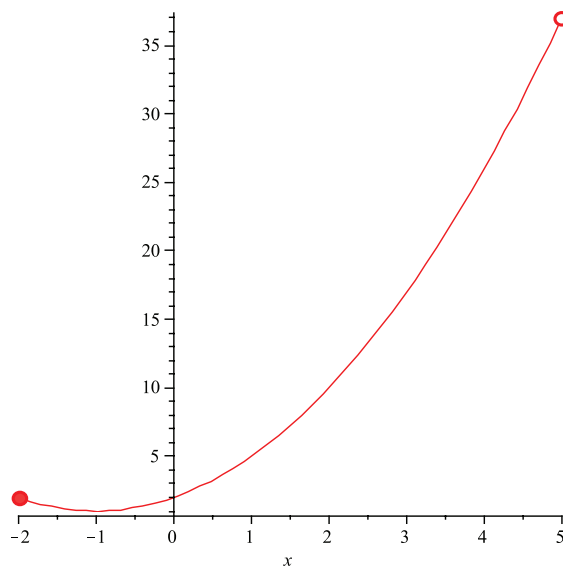
§4.1, 14b: Sketch the graph of a function that has two local maxima, three local minimum, and seven critical numbers.

Solution:



§4.1, 22: Sketch the graph of the function and use your sketch to find the absolute and local maximum and minimum values of $f(x) = 1 + (x + 1)^2$ on the interval $[-2, 5)$.

Solution: The graph of f will be the graph of $y = x^2$ shifted one unit to the left and one unit up:



We can see from this graph that the function has an absolute minimum at $x = -1$, but no absolute maximum

§4.1, 34: Find the critical numbers of the function $g(t) = |3t - 4|$.

Solution: Notice that $g(t)$ can be written in the following form:

$$g(t) = \begin{cases} 3t - 4 & \text{for } t > 4/3 \\ 4 - 3t & \text{for } t < 4/3. \end{cases}$$

So we see that $g'(t) = 3$ for $t > 4/3$ and $g'(t) = -3$ for $t < 4/3$. However, g' is not defined at $t = 4/3$, and so $t = 4/3$ is our only critical point.

§4.3, 6: The graph of a derivative f' of a function f is shown. On what intervals is f increasing or decreasing, and at what values of x does f have a local maximum or minimum?

Solution:

- (a) f is increasing on the intervals $(0, 1)$, and $(3, 5)$, where f' is positive. f is decreasing on the intervals $(1, 3)$ and $(5, 6)$, where f' is negative.
- (b) The critical points of f are $x = 1, 3, 5$, where $f'(x) = 0$. f has a local max at $x = 1, 5$, since f' is going from positive to negative there, and f has a local min at $x = 3$, since f' is going from negative to positive there.

§4.3, 8: The graph of the first derivative f' of a function f is shown. On what intervals is f increasing? At what values of x does f have a local maximum or minimum? On what intervals is f concave up or concave down? What are the inflection points of f ?

Solution:

- (a) The function is increasing on $(2, 4)$ and $(6, 9)$, since the derivative is positive there.
- (b) f has a local maximum at $x = 4$, since $f'(4) = 0$, and f' goes from being positive to negative there. f will have a local minimum at $x = 2, 6$, since $f'(2) = f'(6) = 0$, and f' goes from being negative to positive at those points.
- (c) f will be concave up on the intervals $(1, 3)$, $(5, 7)$, and $(8, 9)$, since f' is increasing on these intervals. f will be concave down on the intervals $(0, 1)$, $(3, 5)$, and $(7, 8)$, since f' is decreasing on these intervals.
- (d) Inflection points will occur at the critical points of f' , or at $x = 1, 3, 5, 7, 8$.

§4.3, 20: Find the local maximum and minimum values of $f(x) = \frac{x}{x^2+4}$ using both the first and second derivative tests. Which method do you prefer?

Solution:

$$f'(x) = \frac{(x^2 + 4) - x(2x)}{(x^2 + 4)^2} = \frac{4 - x^2}{(x^2 + 4)^2}.$$

So $f'(x)$ is defined everywhere, and equals zero at $x = \pm 2$.

We first apply the first derivative test. Notice that $f'(x) > 0$ for $-2 < x < 2$ and $f'(x) < 0$ for $x < -2$ and $x > 2$. So at $x = -2$, f' goes from being negative to positive, so f has a local minimum at $x = -2$. At $x = 2$, f' goes from being positive to negative. so f has a local max at $x = 2$.

We next apply the second derivative test. The second derivative of f is:

$$f''(x) = \frac{-2x(x^2 + 4)^2 - (4 - x^2) \cdot 2(x^2 + 4)(2x)}{(x^2 + 4)^4}.$$

Notice:

$$f''(-2) = \frac{4(8)^2}{8^4} > 0.$$

So f has a local min at $x = -2$. Also notice:

$$f''(2) = \frac{-4(8)^2}{8^4} < 0.$$

So f has a local max at $x = 2$.

§4.3, 53: Suppose the derivative of the function f is $f'(x) = (x + 1)^2(x - 3)^5(x - 6)^4$ on what interval is f increasing?

Solution: The function f will be increasing anywhere f' is positive. Notice that the factors $(x + 1)$ and $(x - 6)$ are both raised to even powers, and so will be nonnegative or all values of x . The $(x - 3)$ factor, however, is raised to an odd power, and so will only be positive where $x > 3$.

§4.2, 66: Consider the function $f(x) = e^{-x^2/2\sigma^2}$. Find the asymptote, maximum value, and inflection points of f . What role does σ play in the shape of the curve? Illustrate by graphing four members of the family on the same screen.

Solution:

- (a) First, we can see that as x goes to positive or negative infinity, f goes to zero. So f has a horizontal asymptote at $y = 0$.

Differentiating f with respect to x gives us:

$$f'(x) = e^{-x^2/2\sigma^2} \frac{-x}{\sigma^2}$$

which will be zero where $x = 0$. So f has a critical point at $x = 0$. So $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$. Thus, f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. So the absolute maximum of f is $f(0) = 1$.

Differentiating f' again gives us:

$$\begin{aligned} f''(x) &= \frac{-e^{-x^2/2\sigma^2}}{\sigma^2} + e^{-x^2/2\sigma^2} \cdot \frac{x^2}{\sigma^4} \\ &= e^{-x^2/2\sigma^2} \left(-\frac{1}{\sigma^2} + \frac{x^2}{\sigma^4} \right). \end{aligned}$$

So $f''(x) = 0$ at $x = \pm\sigma$.

(b) We can see that f can be written as

$$f(x) = e^{-(x/\sigma\sqrt{2})^2}.$$

So the $(x/\sigma\sqrt{2})^2$ corresponds to a horizontal stretch of the graph of $y = e^{-x^2}$. The σ here affects the degree of this stretching.

(c) Below, the red curve represents $\sigma = 1$, the green curve represents $\sigma = 2$, the blue curve represents $\sigma = 4$, and the yellow curve represents $\sigma = 1/2$.

