CALCULUS HOMEWORK SOLUTIONS
WEEK 8

§3.9, 27: Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min, and its coarseness is such that is forms a pile in the shape of a cone whose diameter and height are always equal. How fast is the pile increasing when the pile is 10 ft high?

Solution: The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. Since the height and diameter are the same, this formula becomes

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{3} \cdot \frac{h^3}{4} = \frac{\pi}{12} h^3$$

Differentiating this gives us:

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

We are told that $\frac{dV}{dt} = 30\ \text{ft}^3/\text{min}$. So when $h = 10$ ft, we have

$$30 = \frac{\pi}{4} \cdot (100) \frac{dh}{dt}.$$

Solving for $\frac{dh}{dt}$ gives us

$$\frac{dh}{dt} = \frac{6}{5\pi}.$$

§3.9, 34: Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimeters) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved form 15 cm to 20 cm at a constant rate, how fast was this species' brain growing when the average length was 18 cm?

Solution: Since B is a function of W, W is a function of L and L is a function of time, then

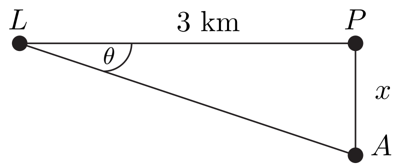
$$\frac{dB}{dt} = \frac{dB}{dW} \cdot \frac{dW}{dL} \cdot dLdt = [0.007(2/3)W^{-1/3}][.12(2.53)L^{1.53}]\frac{dL}{dt}.$$

Letting $L = 18$ cm and $\frac{dL}{dt} = 1/2$ cm per million years, we get

$$\frac{dB}{dt} = [0.007(2/3)(0.12(18)^{2.53})^{-1/3}][0.12(2.53)(18)^{1.53}][1/2] \approx .01045.$$

So when $L = 18$cm, $\frac{dB}{dt} = .01045$ grams per million years.

§3.9, 38: A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the light moving along the shoreline when it is 1 km from P?



Solution: In the above graph, let L represent the lighthouse, and A be the point 1 km away from P. We need to find $\frac{dx}{dt}$ when $x = 1$ km. We are told that the light makes four revolutions per minute. Since each revolution is 2π radians, this means that the light is turning at 8π radians per minute. So $\frac{d\theta}{dt} = 8\pi$.

Notice $\tan\theta = \frac{x}{3}$. Differentiating both sides with respect to time gives us

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{3} \cdot \frac{dx}{dt}.$$

Notice that when $x = 1$, $\sec^2\theta = \frac{10}{9}$. With this in mind, we can solve the above equation for $\frac{dx}{dt}$:

$$\frac{dx}{dt} = 3 \cdot \frac{10}{9} \cdot 8\pi = \frac{80\pi}{3}\text{km/min}.$$

§3.10, 2: Find the linearization $L(x)$ of the function $f(x) = \ln(x)$ at $a = 1$.

Solution:

$$L(x) = f(a) + f'(a)(x-a) = \ln(1) + \frac{1}{1}(x-1) = (x-1).$$

§3.10, 5: Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{.9}$ and $\sqrt{.99}$. Illustrate by graphing f and the tangent line.

Solution:

$$L(x) = f(a) + f'(a)(x-a) = \sqrt{1-a} - \frac{1}{2\sqrt{1-a}}(x-a) = 1 - \frac{x}{2}.$$

To approximate $\sqrt{0.9}$, we need to find x such that $f(x) = \sqrt{0.9}$. To that end, notice $\sqrt{0.9} = \sqrt{1-0.1}$, so our x here is 0.1. Pluggins this into $L(x)$ gives us

$$L(0.1) = 1 - \frac{0.1}{2} = .95.$$

Similarly, to approximate $\sqrt{.99}$, we plug $x = 0.01$ into $L(x)$:

$$L(0.01) = 1 - \frac{0.01}{2} = .995.$$

§3.10, 12: Find the differential of the function $y = s/(1+2s)$.

Solution: Differentiating both sides with respect to s gives us:

$$\frac{dy}{ds} = \frac{1}{(1+2s)^2}$$

From this, we see that the differential of the function is:

$$dy = \frac{1}{(1+2s)^2} ds.$$

§3.10, 28: Use a linear approximation or differentials to estimate $\sqrt{99.8}$.

Solution: Using a linear approximation: We will use the linearization of the function $f(x) = \sqrt{x}$ at the point $x = 100$. This gives us:

$$L(x) = f(a) + f'(a)(x-a) = 10 + \frac{1}{20}(x-100).$$

Plugging $x = 99.8$ into this gives us

$$L(99.8) = 10 - \frac{.2}{20} = 9.99.$$

Thus,

$$\sqrt{99.8} \approx 9.99.$$

Using differentials: We can use the function $y = \sqrt{x}$, and differentiate to get:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

which gives us the differential:

$$dy = \frac{1}{2\sqrt{x}} dx.$$

Where $x = 100$ and $dx = -0.2$, we get

$$dy = \frac{1}{2\sqrt{100}}(-0.2) = -0.01.$$

So we have

$$\sqrt{99.8} = f(99.8) \approx f(100) + dy = 10 - 0.01 = 9.99$$

Therefore,

$$\sqrt{99.8} \approx 9.99.$$