CALCULUS HOMEWORK SOLUTIONS
WEEK 7

§3.5, 36: Use implicit differentiation to find y'' of

$$x^4 + y^4 = a^4.$$

Solution: The first round of implicit differentiation gives us:

$$4x^3 + 4y^3y' = 0$$

since a is a constant. Solving this for y' gives us

$$y' = -\frac{x^3}{y^3}.$$

Differentiating again yields:

$$y'' = -\frac{3x^2y^3 - 3y^2x^3y'}{y^6}.$$

Substituting for y' gives us

$$y'' = -\frac{3x^2y^3 + 3y^2x^3\frac{x^3}{y^3}}{y^6}$$

which simplifies to

$$y'' = -\frac{3x^2y^4 + 3x^6}{y^7}$$

§3.5, 39: Find the points on the graph

(1) $$2(x^2 + y^2)^2 = 25(x^2 - y^2)$$

where the tangent line is horizontal.

Solution: Using implicit differentiation gives us

$$4(x^2 + y^2)(2x + 2yy') = 25(2x - 2yy').$$

Setting $y' = 0$ yields:

(2) $$8x(x^2 + y^2) = 50x.$$

We can see this equation holds where $x = 0$. If $x \neq 0$, then this simplifies to

$$x^2 + y^2 = \frac{25}{4}.$$

Solving this for y^2 gives us

(3) $$y^2 = \frac{25}{4} - x^2.$$

In order to find the points on curve [1] where the tangent line is zero, we need to find all the points (x, y) that satisfy both equation [1] and equation [2]. We saw before that $x = 0$ satisfies equation

[2]. So setting $x = 0$ in equation [1] yields:

$$2y^4 = -25y^2$$

which will hold for $y = 0$. However, we can see from the graph of equation [2], given in exercise 29, that the derivative at the point (0,0) does not exist. So (0,0) is not one of our desired points.

Now, we plug equation [3] into equation [1] and get:

$$2(x^2 + \frac{25}{4} - x^2)^2 = 25(x^2 - \frac{25}{4} + x^2)$$

Which simplifies to

$$\frac{25}{16} = x^2 - \frac{25}{8}.$$

So $x^2 = \frac{75}{16}$, or $x = \pm\frac{5\sqrt{3}}{4}$. Substituting $x^2 = \frac{75}{16}$ into equation [3] gives the corresponding y values:

$$y^2 = \frac{25}{4} - \frac{75}{16} = \frac{25}{4}.$$

So $y = \pm\frac{5}{4}$. So the curve has a horizontal tangent line at four points: $(\frac{5\sqrt{3}}{4}, \pm\frac{5}{4})$ and $(-\frac{5\sqrt{3}}{4}, \pm\frac{5}{4})$

§3.5, 60: Show the following two sets of curves are orthogonal trajectories of each other, and sketch both families of curves on the same axes:

$$x^2 + y^2 = ax, \qquad x^2 + y^2 = by.$$

Solution: For our purposes, we will assume a and b are both nonzero (otherwise, both curves will consist of just the origin). Furthermore, we can see that both curves are circles, the first being of radius $a/2$ centered at $(a/2, 0)$ and the second being of radius $b/2$ and centered at $(0, b/2)$. That said, we can see they meet orthogonally at the origin.

Now, assuming neither x or y are zero. The derivative of the first curve is given by

$$2x + 2yy_1' = a$$

and thus $y_1' = \frac{a-2x}{2y}$. The derivative of the second curve is given by:

$$2x + 2yy_2' = by_2'$$

and thus $y_2' = \frac{2x}{b-2y}$. So if the two curves meet at the point (x_0, y_0), we need to prove

$$y_1'(x_0, y_0) = -\frac{1}{y_2'(x_0, y_0)}.$$

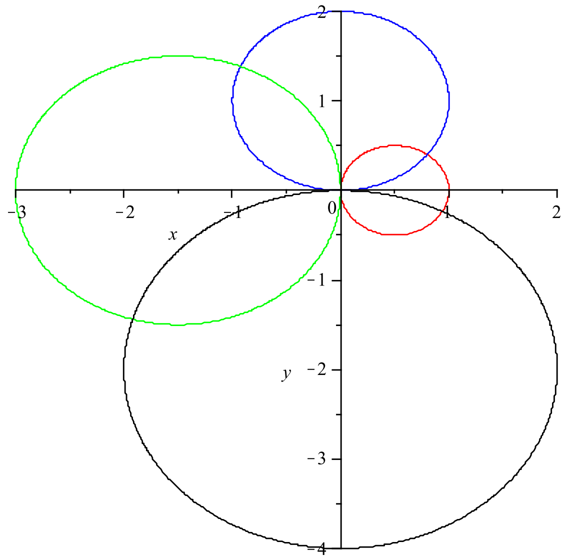
We will do this by showing $y_1'(x_0, y_0) + \frac{1}{y_2'(x_0,y_0)} = 0$. Notice:

$$\begin{aligned} y_1'(x_0, y_0) + \frac{1}{y_2'(x_0, y_0)} &= \frac{a - 2x_0}{2y_0} + \frac{b - 2y_0}{2x_0} \\ &= \frac{(a - 2x_0)x_0 + (b - 2y_0)y_0}{2x_0y_0} \\ &= \frac{ax_0 - 2x_0^2 + by_0 - 2y_0^2}{2x_0y_0} \\ &= \frac{ax_0 - x_0^2 - y_0^2 + by_0 - x_0^2 - y_0^2}{2xy}. \end{aligned}$$

Since (x_0, y_0) is on both curves, we know it satisfies the equation $x_0^2 + y_0^2 = ax_0$. Thus, we see that $ax_0 - x_0^2 - y_0^2 = 0$. Similarly, we get that $by_0 - x_0^2 - y_0^2 = 0$. With this in mind, we see that

$$\frac{ax_0 - x_0^2 - y_0^2 + by_0 - x_0^2 - y_0^2}{2xy} = \frac{0}{2xy} = 0.$$

Therefore, $y_1'(x_0, y_0) = -\frac{1}{y_2'(x_0,y_0)}$, so the two curves are orthogonal at the point (x_0, y_0). Since (x_0, y_0) were arbitrarily chosen, then we can say the two curves will be orthogonal anywhere they meet. Therefore, the two families of curves are orthogonal trajectories.



§3.6, 30: Differentiate and find the domain of

$$f(x) = \ln(\ln(\ln(x))).$$

Solution: We will start by finding the domain of f. First, recall that $\ln(x)$ is not defined for all $x \leq 0$. So the domain of f will be all the points where $\ln(\ln(x)) > 0$. The will occur where $\ln(x) > 1$, or where $x > e$. so the domain of f is (e, ∞).

$$\begin{aligned}
\frac{d}{dx}[f(x)] &= \frac{d}{dx}[\ln(\ln(\ln(x)))] \\
&= \frac{1}{\ln(\ln(x))} \cdot \frac{d}{dx}[\ln(\ln(x))] \\
&= \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \cdot \frac{d}{dx}[\ln(x)] \\
&= \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}.
\end{aligned}$$

§3.6, 48: Use logarithmic differentiation to find the derivative of the function
$$y = (\ln(x))^{\cos(x)}.$$

Solution:
$$\begin{aligned} y = (\ln(x))^{\cos(x)} &\Rightarrow \ln(y) = \cos(x)\ln(\ln(x)) \\ &\Rightarrow \frac{y'}{y} = -\sin(x)\ln(\ln(x)) + \frac{\cos(x)}{x\ln(x)} \\ &\Rightarrow y' = y\left[-\sin(x)\ln(\ln(x)) + \frac{\cos(x)}{x\ln(x)}\right] \\ &\Rightarrow y' = (\ln(x))^{\cos(x)}\left[-\sin(x)\ln(\ln(x)) + \frac{\cos(x)}{x\ln(x)}\right]. \end{aligned}$$

§3.7, 12:

a) Let V be the volume of a cube with side length x. Calculate dV/dx when $x = 3mm$ and explain its meaning.
b) Show that the rate of change of the volume of a cube with respect to its edge length is equal to half the surface area of the cube.

Solution:

a) Since $V(x) = x^3$, then $\frac{dV}{dx}(x) = 3x^2$. Thus, $\frac{dV}{dx}(3) = 27$. So when the side length is 3, the ratio of change in volume to change in side length is $27\text{mm}^3/\text{mm}$.
b) Notice that
$$\frac{dV}{dx}(x) = 3x^2 = \frac{1}{2}(6x^2) = \frac{1}{2}S(x),$$
where $S(x)$ is the surface area of the cube. This means that for small changes in x, the change in volume is approximately half of the surface area.

§3.7, 13:

Solution: See solutions manual

§3.7, 17:

Solution: See solutions manual

§3.7, 24: The population of yeast cells in a laboratory culture is modeled by the function
$$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$
where t is measured in hours. At time $t = 0$, the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b. According to this model, what happens to the yeast population in the long run?

Solution: Since $f(0) = 20$ we have

$$f(0) = \frac{a}{1 + be^0} = \frac{a}{1+b} = 20$$

and thus $a = 20(1 + b)$. Taking the derivative of f gives us

$$f'(t) = -\frac{abe^{-0.7t}(-0.7)}{(1 + be^{-0.7t})^2}$$

Therefore

$$f'(0) = \frac{ab(0.7)}{(1+b)^2}$$

Since $a = 20(1 + b)$, this simplifies to

$$f'(0) = \frac{20b(0.7)}{1+b} = 12$$

Therefore $14b = 12 + 12b$, and thus $b = 6$ and $a = 20(1 + 6) = 140$.

Notice

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{a}{1 + be^{-0.7t}} = \frac{a}{1 + b \cdot 0} = a$$

So as t goes to infinity, the population stabilizes at $a = 140$ cells.

§3.8, 2: A bacteria cell divides into two cells every 20 minutes. The initial population of a culture is 60 cells.

a) Find the relative growth rate.
b) Find an expression for the number of cells after t hours.
c) Find the number of cells after 8 hours.
d) Find the rate of growth after 8 hours.
e) When will the population reach 20,000 cells?

Solution:

a) We know $P(t) = P(0)e^{kt} = 60e^{kt}$. We also know that after 1/3 of one hour, there will be 120 cells. So $P(\frac{1}{3}) = 60e^{k/3} = 120$. Solving this for k yields

$$e^{k/3} = 2 \Rightarrow k/3 = \ln(2) \Rightarrow k = \ln(8).$$

So the relative growth rate is $k = \ln(8)$.

b) $P(t) = 60e^{t\ln(8)} = 60\left(e^{\ln(8)}\right)^t = (60)8^t$
c) $P(8) = (60)8^8 = 1,006,632,960$
d) We know that $\frac{dP}{dt}(t) = kP(t)$. So $P'(8) = \ln(8)P(8) = \ln(8)(60)8^8 \approx 2.093$ billion cells per hour.
e) $P(t) = 20,000$ where $(60)8^t = 20,000$ or where $t\ln(8) = \ln(1000/3)$. So $P(t) = 20,000$ at $t = \frac{\ln(1000/3)}{\ln(8)} \approx 2.79$ hours.

§3.8, 10: A sample of tritium-3 decayed to 94.5% of its original amount after a year.

a) What is the half-life of tritium-3?
b) How long would it take a sample to decay to 20% of its original amount?

Solution:

a) Let $y(t)$ be the mass of tritium-3 after t years, and $y(0) = M$. So then $y(t) = Me^{kt}$. Since the sample has decayed to 94.5% of its original amount after a year, then $y(1) = Me^k = .945M$. So $k = \ln(.945)$

With that in mind, we need to find the value of t such that $y(t) = \frac{1}{2}M$, which will occur where $e^{\ln(.945)t} = 1/2$, or where $t = \frac{\ln(1/2)}{\ln(.945)}$, or about 12.25 years.

b) As above, we need to find the value of t such that $y(t) = \frac{1}{5}M$. So $t = \frac{\ln(1/5)}{\ln(.945)}$, or about 28.45 years.

§3.8, 16: A freshly brewed cup of coffee has a temperature of 95°C in a 20°C room. When the temperature is 70°C, it is cooling at a rate of 1°C per minute. When does this occur?

Solution: By Newton's Law of Cooling, we know $\frac{dT}{dt} = k(T(t) - 20)$. Letting $y(t) = T - 20$, we get $y'(t) = ky(t)$. Notice $y(0) = T(0) - 20 = 95 - 20 = 75$. Thus, $y(t) = 75e^{kt}$. We know that when $T(t) = 70$, $dT/dt = -1$. So when $y(t) = 50$, $y'(t) = -1$. So $y'(t) = ky(t) = 50k = -1$. So $k = -1/50$. Further, $y(t) = 50 = 75e^{-t/50}$ This gives us $e^{-t/50} = 2/3$, so $t = -50\ln(2/3) \approx 20.27$ minutes.