CALCULUS HOMEWORK SOLUTIONS
WEEK 12

§5.1, 5:

a) Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using three rectangles and right end-points. Then, improve your estimate by using six rectangles. Sketch the curve and the approximating triangles.
b) Repeat part (a) using left end-points
c) Repeat part (a) using midpoints.
d) From the sketches of (a)-(c), which appears to be the best estimate?

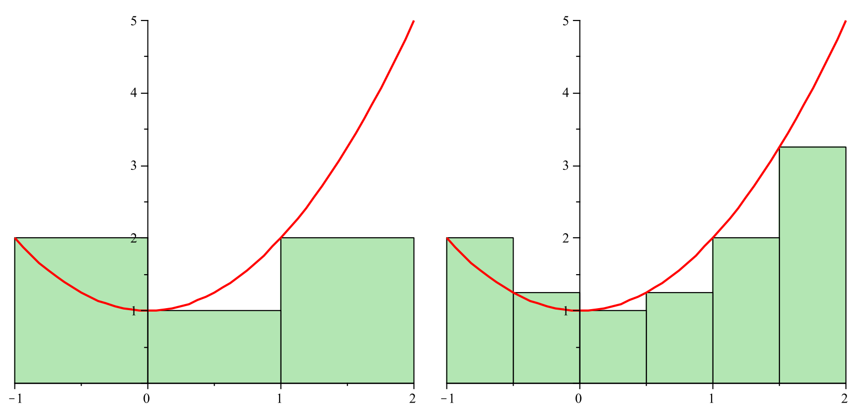
Solution:

a) Using three rectangles, we have $\Delta x = \frac{2-(-1)}{3} = 1$. Using right end-points, we have

$$R_3 = 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2) = 1 + 2 + 5 = 8.$$

Using six rectangles, we have $\Delta x = \frac{1}{2}$.

$$\begin{aligned} R_6 &= \frac{1}{2}(f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)) \\ &= \frac{1}{2}(2 + 1.25 + 1 + 1.25 + 2 + 3.25) \\ &= 6.875. \end{aligned}$$

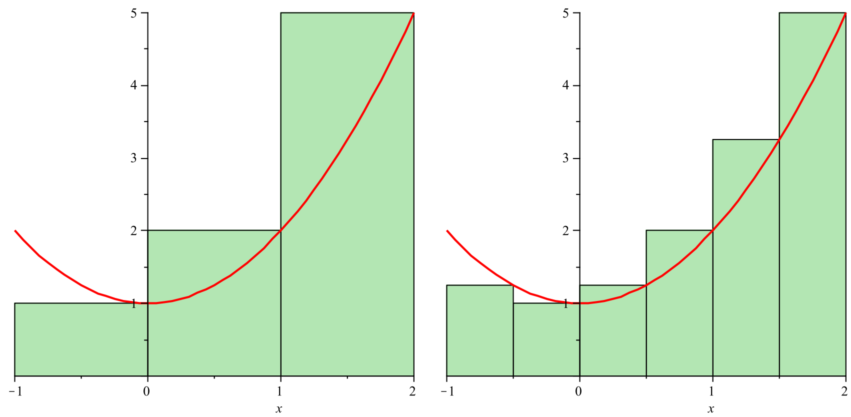


b) Again, using three rectangles, we have $\Delta x = 1$, and using left endpoints, we get:

$$L_3 = 1(f(-1) + f(0) + f(1)) = 2 + 1 + 2 = 5.$$

Using six rectangles we have $\Delta x = .5$ and

$$\begin{aligned} L_6 &= \frac{1}{2}(f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5)) \\ &= \frac{1}{2}(2 + 1.25 + 1 + 1.25 + 2 + 3.25) \\ &= 5.375. \end{aligned}$$

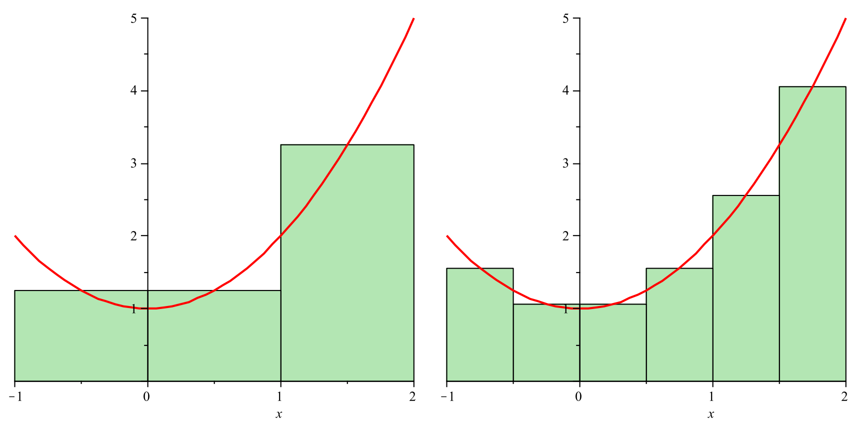


c) Using three rectangles and midpoints, we get

$$M_3 = 1(f(-0.5) + f(0.5) + f(1.5) = 1.25 + 1.25 + 3.25 = 5.75.$$

Using six rectangles and midpoints, we get:

$$\begin{aligned} M_6 &= .5(f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)) \\ &= .5(1.5625 + 1.0625 + 1.5625 + 2.5625 + 4.0625) \\ &= 5.9375 \end{aligned}$$



d) Looking at each figure, M_6 looks like the best estimate.

§5.1, 12: Speedometer readings for a motorcycle at 12-second intervals are given in the table.

a) Estimate the distance traveled by the motorcycle during this time period using the velocities at the beginning of the time intervals.
b) Give another estimate using the values at the end of the time periods.
c) Are the estimates in parts (a) and (b) upper and lower estimates? Explain.

Solution:

a) Using the velocities at the beginning of the time periods, we get
$$d_L = 12 \cdot 30 + 12 \cdot 28 + 12 \cdot 25 + 12 \cdot 22 + 12 \cdot 24 = 1548\text{ft}.$$
b) Using the velocities at the end of the time periods, we get
$$d_R = (28 + 25 + 22 + 24 + 27) \cdot 12 = 1512\text{ft}.$$
c) The estimates in parts (a) and (b) are neither upper or lower estimates, since velocity is neither an increasing nor decreasing function of t

§5.1, 14: Use the data on the given table to estimate the height above the earth's surface of the *Endeavour*, 62 seconds after liftoff.

Solution: We first find a lower bound for the height of the shuttle, by using the initial velocity for each interval. So the distance traveled after 62 seconds could be estimated by:
$$\begin{aligned} d_\ell &= \sum_{i=1}^{6} v(t_i)\Delta t_i \\ &= 0 \cdot 10 + 185 \cdot 5 + 319 \cdot 5 + 447 \cdot 12 + 742 \cdot 27 + 1325 \cdot 3 \\ &= 31,893 \text{ ft.} \end{aligned}$$
We can also find an upper bound, using the final velocity for each time interval:
$$\begin{aligned} d_u &= \sum_{i=1}^{6} v(t_i)\Delta t_i \\ &= 185 \cdot 10 + 319 \cdot 5 + 447 \cdot 5 + 742 \cdot 12 + 1325 \cdot 27 + 1445 \cdot 3 \\ &= 54,694 \text{ ft.} \end{aligned}$$

§5.1, 18: Use Definition 2 to find an expression for the area under the graph of
$$f(x) = \frac{\ln(x)}{x}, \quad 3 \leq x \leq 10$$
as a limit. Do not evaluate the limit.

Solution: $\Delta x = \frac{10-3}{n} = \frac{7}{n}$, and $x_i = 3 + i\Delta x = 3 + \frac{7i}{n}$. Then
$$A = \lim_{n\to\infty} \sum_{n\to\infty}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{n\to\infty}^{n} \frac{\ln(3 + \frac{7i}{n})}{3 + \frac{7i}{n}} \cdot \frac{7}{n}.$$

§5.1, 20: Determine a region whose area is equal to

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left(5 + \frac{2i}{n}\right)^{10}.$$

Do not evaluate the limit.

Solution: From what we're given, we have $\Delta x = \frac{2}{n}$, and $x_i = 5 + \frac{2i}{n}$. So the interval we're looking has length 2. If our interval starts as $x = 5$, then our function is $y = x^{10}$ on the interval [5,7]. So a region whose area gives the desired sum is the region under the curve $y = x^1 0$ over the interval $[5, 7]$.

This answer, however, is not unique. In general, we could have $y = ((5-k)+x)^{10}$ on the interval [k,k+2] for any number k. Notice $\Delta x = \frac{k+2-k}{n} = \frac{2}{n}$, and $x_i = k + i\Delta x = (k + \frac{2i}{n})$.

§5.1, 22:

a) Use Definition 2 to find an expression for the area under the curve $y = x^3$. from 0 to 1 as a limit.

b) Use the following formula for the sum of the cubes of the first n integers:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

to evaluate the limit in part (a).

Solution:

a) $\Delta x = \frac{1-0}{n} = \frac{1}{n}$, then $x_i = 0 + i\Delta x = \frac{i}{n}$. Then

$$A = \lim_{R\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n}.$$

b) Using Use the following formula for the sum of the cubes of the first n integers, we get

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n} &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^3} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 \\
&= \lim_{n\to\infty} \frac{1}{n^4} \left[\frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2} \\
&= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right) \\
&= \frac{1}{4}.
\end{aligned}$$