MATH 200X
Calculus I
Fall 2006

Exam I (Solutions)

1. Evaluate the following limits.

   (a) $\lim_{x\to 3} \frac{\sqrt{3}-\sqrt{x}}{3-x} = \lim_{x\to 3} \frac{\sqrt{3}-\sqrt{x}}{(\sqrt{3}-\sqrt{x})(\sqrt{3}+\sqrt{x})} = \lim_{x\to 3} \frac{1}{\sqrt{3}+\sqrt{x}} = \frac{1}{2\sqrt{3}}$

   (b) $\lim_{x\to 4^+} \frac{|4-x|}{4-x} = \lim_{x\to 4^+} \frac{-(4-x)}{4-x} = \lim_{x\to 4^+} -1 = -1$

2. Evaluate the following limits at infinity.

   (a) $\lim_{x\to\infty} \frac{2-3x}{\sqrt{5x^2+1}} = \lim_{x\to\infty} \frac{(2-3x)\frac{1}{x}}{(\sqrt{5x^2+1})\frac{1}{x}} = \lim_{x\to\infty} \frac{2/x-3}{\sqrt{5+1/x^2}} = -3/\sqrt{5}$

   (b) $\lim_{x\to-\infty} \ln\left(\frac{-1}{x}\right) = -\infty$
   This follows from the fact that $\lim_{x\to-\infty}(-1/x) = 0^+$. And we know from the graph of $\ln x$ that $\lim_{x\to 0^+}(\ln x) = -\infty$.

3. Let $f(x) = \frac{2x-2}{x^2+4x-5}$.

   (a) What is the domain of $f(x)$?
   $x^2+4x-5 = (x+5)(x-1) = 0$ when $x = 1$ or $x = -5$
   ANSWER: All real numbers except 1 and -5.

   (b) Find any vertical asymptotes. Show your work.
   $\lim_{x\to-5^+} \frac{2x-2}{x^2+4x-5} = -\infty$. So $x = -5$ is an asymptote.
   $\lim_{x\to 1^+} \frac{2x-2}{x^2+4x-5} = \lim_{x\to 1^+} \frac{2(x-1)}{(x+5)(x-1)} = 1/3$. So there is no asymptote at $x = 1$.

4. (a) Complete the following definition:
   A function f is continuous at a number a if (look in your book.)

   (b) Use the definition to determine if the function defined below is continuous at $x = 5$. You must show your work.

   $$f(x) = \begin{cases} \frac{x-3}{1+x} & x < 5 \\ \frac{1}{3}(7-x) & x \geq 5 \end{cases}$$

   The limit of $f(x)$ as x approaches 5 does not exist since the left-hand limit is 1/3 and the right hand limit is 2/3. Since the limit does not exist, the function is not continuous at $x = 5$.

5. (a) Complete the following definition: The derivative of a function $f(x)$ with respect to the variable x is the function $f'(x)$ defined as:
   (look in your book.)

   (b) Use the definition to find $f'(x)$ for $f(x) = 3x - 6x^2$. You must show your work.
   $f'(x) = \lim_{h\to 0} \frac{3(x+h)-6(x+h)^2-[3x-6x^2]}{h} = \lim_{h\to 0} \frac{3x+3h-6x^2-12xh-h^2-3x+6x^2}{h} = \lim_{h\to 0} \frac{3h-12xh-h^2}{h} =$
   $\lim_{h\to 0} 3 - 12x - h = 3 - 12x$.

   (c) Find the equation of the line tangent to $f(x)$ at $x = 1$.
   $f(1) = -3$; So the point $(1, -3)$ is on the tangent.
   $f'(1) = -9$; So the slope, m, of the tangent is -9.
   Thus the equation of the line is: $y - (-3) = -9(x-1)$ or equivalently $y = -9x + 6$.

6. A sociologist is studying an educational program for preschool-age children in a certain city. He lets the variable t represent the number of years after the beginning of the program. He then produces a formula $f(t)$ estimating the number of children enrolled in the program where $f(t)$ is measured in thousands preschoolers enrolled.

   (a) Explain what $f(2) = 5$ means and determine its units.
   $f(2) = 5$ means that 2 years after the program has begun, 5 thousand students are enrolled.
   units of $f(x)$: thousands of students

   (b) Explain what $f'(2) = 1$ means and determine its units.
   $f'(2) = 1$ means that two years after the program has started, the instantaneous rate of change of enrolled students with respect to time is one thousand. Or another way: At two years, the rate of change of students per year is 1 thousand students. Or a third way: At two years, you expect that the program added approximately 1 thousand students from last year and you expect to add about 1 thousand students in the next year.
   units of $f'(x)$: thousands of students/year.

7. Consider the following graph of $f(x)$. see your paper

   (a) What is $\lim_{x \to 0^-} f(x)$?
   1

   (b) What is $\lim_{x \to 0^+} f(x)$?
   ∞

   (c) What is $\lim_{x \to 2^-} f(x)$?
   3

   (d) What is $\lim_{x \to \infty} f(x)$?
   ∞

   (e) What is $\lim_{x \to -\infty} f(x)$?
   1

   (f) For what values of a does $\lim_{x \to a} f(x)$ fail to exist?
   $a = 0$ and $a = 2$

   (g) For what values of a does $f(a)$ fail to be continuous?
   $a = 0, 2$, and 4

   (h) For what values of a does $f(a)$ fail to be differentiable?
   $a = 0, 2$, and 4