3.1 #28

Differentiate the function $f(v) = ae^v + \frac{b}{v} + \frac{c}{v^2}$ where a, b and c are constants.

Solution: Notice $f(v) = ae^v + bv^{-1} + cv^{-2}$. Then using the sum rule, power rule, constant multiple rule and the derivative of e we have $\frac{d}{dv}f(v)$ is,

$$\begin{aligned}
\frac{d}{dv}f(v) &= \frac{d}{dv}(ae^v + bv^{-1} + cv^{-2}) \\
&= \frac{d}{dv}ae^v + \frac{d}{dv}bv^{-1} + \frac{d}{dv}cv^{-2} \\
&= a\frac{d}{dv}e^v + b\frac{d}{dv}v^{-1} + c\frac{d}{dv}v^{-2} \\
&= ae^v + b(-1)v^{-2} + c(-2)v^{-3} \\
&= ae^v - \frac{b}{v^2} - \frac{2c}{v^3}.
\end{aligned}$$

3.1 #60

a. Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $f(x) = x^2 + x$.

Solution: We first find the derivative of $f(x)$ using the power rule we have $f'(x) = 2x + 1$. Then the point slope equation of a line we have $y+3 = m(x-2)$. Since $f'(x) = 2x+1$ we take $m = 2x+1$. This gives us the equation $y+3 = (2x+1)(x-2)$. Now we notice that $y = x^2 + x$. Then $x^2 + x + 3 = (2x+1)(x-2)$, or $x^2 + x + 3 = 2x^2 - 3x - 2$, or $0 = x^2 - 4x - 5$. But this factors into $0 = (x-5)(x+1)$. Therefore $x = 5$ and $x = -1$.

When $x = 5$ we have the slope of the tangent line $m = 2(5) + 1 = 11$. Then the equation is given by $y + 3 = 11(x - 2)$, or $y = 11x - 25$. When $x = -1$ we have the slope of the tangent line $m = 2(-1) + 1 = -1$. Then the equation of the line is given by $y + 3 = (-1)(x - 2)$ or $y = -x - 1$. So equations of both lines through the point $(2, -3)$ tangent to $f(x) = x^2 + x$ are given by,

$$y = 11x - 25$$

and

$$y = -x - 1.$$

b. Show that there is no line through the point $(2, 7)$ that is tangent to the parabola.

Solution: If we proceed as above, we have $y-7 = (2x+1)(x-2)$ or $y-7 = 2x^2-3x-2$ or $y = 2x^2 - 3x + 5$. Since $y = x^2 + x$ we have $x^2 + x = 2x^2 - 3x + 5$, or $0 = x^2 - 4x + 5$. Using the quadradic formula we find the roots,

$$y = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2} = \frac{4 \pm \sqrt{-4}}{2}.$$

Since this equation has no real roots, there is no tangent line though the point $(2,7)$ that is tangent to the parabola.

If we draw a picture, you see that the point $(2,7)$ lies inside the parabola. Therefore line though this point will intersect the parabola in two places, and not be tangent.

3.1 #68

At what numbers is the following function g differentiable.

$$g(x) = \begin{cases} -1-2x & \text{if} \quad x < -1 \\ x^2 & \text{if} \quad -1 \le x \le 1 \\ x & \text{if} \quad x > 1 \end{cases}$$

Solution: Notice first that since $-1-2x$, x^2 and x are all differentiable for all real numbers, g is differentiable on $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$. We can also show that g is continuous at -1 and 1 since the left and right limits are equal, and equal to $f(1)$ and $f(-1)$.

Taking derivatives we have,

$$g'(x) = \begin{cases} -2 & \text{if} \quad x < -1 \\ 2x & \text{if} \quad -1 \le x \le 1 \\ 1 & \text{if} \quad x > 1 \end{cases}$$

At $x=-1$ we have that $g'(-1) = -2$ in both the case when $x < -1$ and $x > -1$. Therefore g is differentiable at $x=-1$. At $x=1$ we have $g'(1) = 2$ for $x < 1$ and $g'(1) = 1$ for $x > 1$. Since these values are not equal, g is not differentiable at $x = 1$.

Since x = 1 is the only point where g is not differentiable, we have g is differentiable on $(-\infty,1)\cup(1,\infty)$.

3.2 #12

Differentiate $R(t) = (t+e^t)(3-\sqrt{t})$.

Solution: Notice that when we expand the expression above we get $R(t) = 3t - t^{3/2} + 3e^t -$

$\sqrt{t}e^t$. Then

$$\begin{aligned}
\frac{d}{dt}R(t) &= \frac{d}{dt}(3t - t^{3/2} + 3e^t - \sqrt{t}e^t) \\
&= \frac{d}{dt}3t + \frac{d}{dt}(-t^{3/2}) + \frac{d}{dt}3e^t + \frac{d}{dt}(-\sqrt{t}e^t) \\
&= 3\frac{d}{dt}t - \frac{d}{dt}t^{3/2} + 3\frac{d}{dt}e^t - \frac{d}{dt}t^{1/2}e^t \\
&= 3 \cdot 1 - (3/2)t^{(3/2)-1} + 3e^t - (t^{1/2}e^t + (1/2)t^{(1/2)-1}e^t) \\
&= 3 - \frac{3}{2}t^{1/2} + 3e^t - t^{1/2}e^t - \frac{1}{2}t^{-1/2}e^t \\
&= 3 - \frac{3}{2}\sqrt{t} + 3e^t - \sqrt{t}e^t - \frac{e^t}{2\sqrt{t}} \\
&= 3 - \frac{3}{2}\sqrt{t} + e^t(3 - \sqrt{t} - \frac{1}{2\sqrt{t}})
\end{aligned}$$

3.2 #18

Differentiate

$$f(s) = \frac{1}{s + ke^s}.$$

Solution: Using the quotient rule

$$\begin{aligned}
\frac{d}{ds}f(s) &= \frac{(s + ke^s)\frac{d}{ds}1 - 1\frac{d}{ds}(s + ke^s)}{(s + ke^s)^2} \\
&= \frac{(s + ke^s) \cdot 0 - (1 + ke^s)}{(s + ke^s)^2} \\
&= \frac{-1 - ke^s}{(s + ke^s)^2}
\end{aligned}$$

3.2 #42

If $g(x) = x/e^x$, find $g^{(n)}(x)$.

Solution: To find a formula for the n^{th} derivative we begin by computing the first three derivatives using the chain rule,

$$\begin{aligned}
f'(x) &= \frac{1}{e^x} - \frac{x}{e^x} \\
f^{(2)}(x) &= \frac{-2}{e^x} + \frac{x}{e^x} \\
f^{(3)}(x) &= \frac{3}{e^x} - \frac{x}{e^x}.
\end{aligned}$$

Observe that we always have the two terms n/e^x and x/e^x, where n is the number of derivatives we've taken. Notice the term n/e^x is negative when n is even and positive when n is odd. Then we have to have a coeffient which gives (-1) when raised to an even power. Notice $(-1)^{n+1}$ works. We have that the coefficent of x/e^x is positive when n is even, we have a coefficient of $(-1)^n$.

Then

$$f^{(n)}(x) = \frac{(-1)^{n+1}n}{e^x} + \frac{(-1)^n x}{e^x}.$$

3.3 # 8

Differentiate $y = e^u(\cos(u) + cu)$. Notice

$$\begin{aligned}\frac{dy}{du} &= e^u \frac{d}{du}(\cos(u) + cu) + (\cos(u) + cu)\frac{d}{du}e^u \\ &= e^u(-\sin(u) + c) + (\cos(u) + cu)e^u \\ &= e^u(\cos(u) - \sin(u) + cu + c).\end{aligned}$$

3.3 # 36 a, c, d, e

An elastic band is hung on a book and a mass in hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2\cos t + 3\sin t$, $t \geq 0$, where s is measured in centimeters and t in seconds. (Take the positive direction to be downward.)

(a) Find the velocity and acceleration at time t.

Solution If we have the position function $f(t) = 2\cos t + 3\sin t$ we know that velocity is the derivative of position, so the velocity function is:

$$f'(t) = -2\sin t + 3\cos t.$$

Acceleration is the derivative of velocity, this gives the acceleration function

$$f''(t) = -2\cos t - 3\sin t.$$

(c) When does the mass pass through the equilibrium position for the first time?

The equilibrium position occurs when acceleration is 0, or in our case $f''(t) = 0$. But then

$$\begin{aligned}0 &= -2\cos t - 3\sin t\\ 2\cos t &= -3\sin t\\ \frac{-2}{3}\cos t &= \sin t\\ \frac{-2}{3} &= \frac{\sin t}{\cos t}\\ \frac{-2}{3} &= \tan t\\ \tan^{-1}(-2/3) &= t.\end{aligned}$$

So $f''(t) = 0$ when $t = \tan^{-1}(-2/3)$, or $t = -0.588$. Since $t \geq 0$ and $\tan^{-1}$ has period π this tells us the first t where the mass is at equilibrium is $t = -0.588 + \pi = 2.5536$.

(d) How far from its equilibrium position does the mass travel?

The mass will be farthest away when the position function is greatest. This occurs when the velocity function is zero, or $f'(t) = 0$. We have $0 = -2\sin t + 3\cos t$. Solving for t we have $3/2 = \tan t$, or $t = \tan^{-1}(3/2)$. This gives $t = 0.9828$. Now, we substitute $t = 0.9828$ into the position function. Then $f(0.9828) = 2\cos(0.9828) + 3\sin(0.9828) = 3.6056$ centimeters.

The equilibrium occurs when $t = 2.5536$, we have $f(t) = 0$.

Then the mass travels $3.6056 - 0 = 3.6056$ centimeters from the equilibrium position.

(e) When is the speed the greatest? *Solution:* Speed is greatest when $f'(t)$ has a maximum value. This occurs when the acceleration is zero, or $f''(t) = 0$. Above we found $f''(t) = 0$ when $t = -0.588 + k\pi$ for $k = 1, 2, 3, \ldots$.