2.7 #12

The graphs in the book give the position function of two runners, A and B, who ran a 100-m finish race and finished in a tie.

a. Describe and compare how the runners run the race.

   Solution: Remember that the derivaitve of position at a point gives the velocity at that point. Also, the derivative is the slope of the tangent line at a particular point. To compare the runners we look at the slopes of the tangent lines to each of the runners position function. Runner A runs the race at a constant speed, whereas runner B begins the race with a slower speed (velocity) and increases his speed as time goes on.

b. As what time is the distance between the runners the greatest?

   Solution: The distances between the runners is greatest when the distance (vertically) between the line descibing runner A's position is farthest from the line describing runner B's position. This occurs at (approximately) $t = 9$ seconds.

c. At which time do they have the same velocity?

   Solution: The runners will have the same velocity when the slopes of the tangent lines to both curves are the same. This occurs at (approximately) $t = 9$.

2.7 #20

Sketch the graph of a function for which $g(0) = g'(0) = 0$, $g'(-1) = -1$, $g'(1) = 3$ and $g'(2) = 1$.

Solution: We won't have a sketch here, but these are the main features your sketch needs to have. First, your graph of $g(x)$ must pass through the point $(0, 0)$. This is what $g(0) = 0$ means. Since $g'(0) = 0$ the slope of the tangent line to of g at the point $x = 0$ needs to have slope 0. Since $g(-1) = -1$ you know that the slope of the tangent line to g at the point $x = -1$ has slope -1. This means that g is decreasing when $x = -1$. But when $x = 1$, $g'(1) = 3$. This means that the slope of the tangent line to $g(x)$ at the point $x = 1$ is 3. Therefore $g(x)$ is increasing quickly at the point $x = 3$. At $x = 2$, $g'(x) = 1$, which means that the function $g(x)$ is still increasing, but not as rapidly as at $x = 1$.

2.7 #28

Find $f'(a)$ where

$$f(x) = \frac{x^2 + 1}{x - 2}.$$

We apply the definition of the derivative at a point $x = a$. Recall,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

We use this definition to try to get h to cancel, so that we're no longer dividing by 0 and can apply our limit laws. This is long, tedious and messy. Be careful.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h\to 0} \Big(\frac{(a+h)^2+1}{(a+h)-2} - \frac{a^2+1}{x-2}\Big)\frac{1}{h} \\
&= \lim_{h\to 0} \Big(\frac{(a+h)^2+1}{(a+h)-2}\frac{(a-2)}{(a-2)} - \frac{a^2+1}{x-2}\frac{((a+h)-2)}{((a+h)-2)}\Big)\frac{1}{h} \\
&= \lim_{h\to 0} \Big(\frac{(a^2+2ah+h^2+1)(a-2) - (a^2+1)(a+h-2)}{((a+h)-2)(a-2)}\Big)\frac{1}{h} \\
&= \lim_{h\to 0} \Big(\frac{(a^2+2ah+h^2+1)(a-2) - (a^2+1)(a+h-2)}{((a+h)-2)(a-2)}\Big)\frac{1}{h} \\
&= \lim\nolimits_{h\to 0} \Big(\tfrac{(a^3+2a^2h+ah^2+a-2a^2-4ah-2h^2-2)-(a^3+a+a^2h+h-2a^2-2)}{((a+h)-2)(a-2)}\Big)\tfrac{1}{h} \\
&= \lim_{h\to 0} \Big(\frac{a^2h-4ah-2h^2-h}{((a+h)-2)(a-2)}\Big)\frac{1}{h} \\
&= \lim_{h\to 0} \frac{a^2-4a-2h-1}{((a+h)-2)(a-2)} \\
&= \frac{a^2-4a-1}{(a-2)(a-2)} \\
&= \frac{a^2-4a-1}{(a-2)^2}.
\end{aligned}$$

2.7 # 46

The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.

a. What is the meaning of the derivative $f'(5)$? What are its units?

   *Solution:*The statement $f'(5)$ is the rate at which the number of bacteria is increasing after 5 hours. The units are given by the change in y over change in x, or number of bacteria/hour.

b. Suppose there is an unlimited amount of space at nutrients for the bacteria. Which do you think it larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited, would that affect your conclusion? Explain.

   Solution: If there is an unlimited amount of nutrients the rate of growth will be faster when $t = 10$ than when $t = 5$. Therefore $f'(10) > f'(5)$.

If there was a limited supply of nutrients more bacteria might die off because of the lack of nutrients. The rate of growth will be higher when the population isn't as large, but slow down as the population gets too large to be sustained by the nutrients. Therefore $f'(5) > f'(10)$.

2.8 # 6

Sketch the graph of f' given the graph of f

Solution: For this problem the exact solution isn't as important as you get some key features in your graph. Your graph need to cross through the origin since the derivative is zero at $x = 0$. For $x < 0$ the graph of f is increasing, so f' is positive and therefore above the x-axis. For $x < 0$ the graph of f is decreasing, so f' is negative and therefore below the x-axis. In summary, the of f' is negative for $x < 0$, goes through the origin and is positive for $x > 0$.

2.8 #20

Find the derivative of the function using the definition of the derivative. State the domain of the function and the domain of the derivative. $f(x) = mx + b$.

Solution: Since f is a linear function its domain is all real numbers, $\mathbb{R}$. We then use the definition of the derivative,

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(m(x+h) + b) - (mx + b)}{h} \\
&= \lim_{h \to 0} \frac{mx + mh + b - mx - b}{h} \\
&= \lim_{h \to 0} \frac{mh}{h} \\
&= \lim_{h \to 0} m \\
&= m.
\end{aligned}$$

In summary, $f'(x) = m$. Since $f'(x) = m$ is a constant function it's domain is all real numbers as well.